Exercise 11.1. Let $T \in \mathrm{Ten}^k(V)$ be a tensor on a real vector space V, and let $T_I = T(E_I)$, defined for any multi-index $I = (i_0, \dots, i_{k-1})$ be the coefficients of T w.r.t. some base $(E_i)_i$ of V. Show that $(\sigma T)_I = T_{\sigma^* I}$ for any permutation $\sigma \in S_k$

Solution. Denote $J = (j_0, \dots, j_{k-1}) = \sigma^* I = (i_{\sigma(0)}, \dots, i_{\sigma(k-1)})$ Then

$$\begin{aligned}(\sigma T)_I &= (\sigma T)(E_{i_0}, \dots, E_{i_{k-1}}) \\ &= T(E_{i_{\sigma(0)}}, \dots, E_{i_{\sigma(k-1)}}) \\ &= T(E_{j_0}, \dots, E_{j_{k-1}}) \\ &= T_{\sigma^* I}\end{aligned}$$

□

Exercise 11.2. Let M be a $\mathcal{C}^{r+1}$ manifold and let ω be a differential k-form on M. We say that ω is $\mathcal{C}^r$ at some point $p \in M$ if the component functions of ω w.r.t. some chart φ (that is defined at p) are $\mathcal{C}^r$ at p. Show that this does not depend on which chart φ we use.

Solution. Take two charts φ, $\widetilde{\varphi}$ that are defined at p. Their coordinate vectors are related by the formula $\frac{\partial}{\partial \widetilde{\varphi}^j} = \sum_i \frac{\partial \varphi^i}{\partial \widetilde{\varphi}^j} \frac{\partial}{\partial \varphi^i}$. Using the transformation law for covariant k-tensors, we have

$$\widetilde{\omega}_J = \sum_{I=(i_0,\dots,i_{k-1}) \in \underline{n}^k} \frac{\partial \varphi^{i_0}}{\partial \widetilde{\varphi}^{j_0}} \cdots \frac{\partial \varphi^{i_{k-1}}}{\partial \widetilde{\varphi}^{j_{k-1}}} \omega_I.$$

If the functions ω_I are $\mathcal{C}^r$ at p, since the functions $\frac{\partial \varphi^i}{\partial \widetilde{\varphi}^j}$ are also $\mathcal{C}^r$ (because the transition map $\varphi \circ \widetilde{\varphi}^{-1}$ is $\mathcal{C}^{r+1}$) we conclude that the functions $\widetilde{\omega}_J$ are $\mathcal{C}^r$ at p. □

Exercise 11.3. The goal of this exercise is to show that the wedge product of alternating covariant tensors in a real vector space V is associative.

(a) If a tensor $T \in \mathrm{Ten}^k V$ is alternating, show that $A(T) = k!\, T$.
(b) For two tensors $S \in \mathrm{Ten}^k V$, $T \in \mathrm{Ten}^\ell V$, show that

$$\begin{aligned}A(A(S) \otimes T) &= k!\, A(S \otimes T) \\ A(S \otimes A(T)) &= \ell!\, A(S \otimes T).\end{aligned}$$

(c) Show that the wedge product of alternating tensors $S \in \mathrm{Alt}^k V$, $T \in \mathrm{Alt}^\ell V$, $R \in \mathrm{Alt}^m V$ is associative:

$$S \wedge (T \wedge R) = S \wedge T \wedge R = (S \wedge T) \wedge R.$$

Solution. See Tu's book "An Introduction to Manifolds", Proposition 3.25. □

Exercise 11.4. For a point $p \in \mathbb{R}^3$ and vectors $v, w \in \mathrm{T}_p\mathbb{R}^3 \equiv \mathbb{R}^3$ we define $\omega|_p(v, w) := \det(p \mid v \mid w)$. Show that ω is a smooth differential 2-form on $\mathbb{R}^3$, and express ω as a linear combination of the elementary alternating 2-forms determined by the standard coordinate chart (x^0, x^1, x^2).

Solution. For each point $p \in \mathbb{R}^3$, the function $\omega|_p(v, w) = \det(p \mid v \mid w)$ is linear on each of its two variables $v, w \in \mathbb{R}^3$, and also alternating, therefore ω is a differential form. The elementary covector fields are $\mathrm{d}x^0$, $\mathrm{d}x^1$, $\mathrm{d}x^2$, and the elementary 2-forms are $\mathrm{d}x^0 \wedge \mathrm{d}x^1$, $\mathrm{d}x^1 \wedge \mathrm{d}x^2$ and $\mathrm{d}x^2 \wedge \mathrm{d}x^0$. The calculation

$$\begin{aligned} \omega|_p(v,w) = \det \begin{pmatrix} p^0 & v^0 & w^0 \\ p^1 & v^1 & w^1 \\ p^2 & v^2 & w^2 \end{pmatrix} = \; & p^0(v^1w^2 - v^2w^1) \\ & + p^1(v^2w^0 - v^0w^2) \\ & + p^2(v^0w^1 - v^1w^0) \end{aligned}$$

shows that

$$\begin{aligned} \omega|_p = \; & p^0\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ & + p^1\,\mathrm{d}x^2 \wedge \mathrm{d}x^0 \\ & + p^2\,\mathrm{d}x^0 \wedge \mathrm{d}x^1. \end{aligned}$$

Thus the component functions of ω are the functions $p \mapsto p^i$ which are smooth. This shows that ω is a smooth 2-form. □

Exercise 11.5 (Some properties of the pullback of differential forms)**.** For $F : M \to N$ a smooth map between smooth manifolds, $\omega \in \Omega^k(N)$, $\beta \in \Omega^\ell(N)$ we have:

(a) $F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta)$.

Solution. Fix $p \in M$ and $X_0, \dots, X_{k+\ell-1} \in \mathrm{T}_pM$. We have to show that

$$F^*(\alpha_{F(p)} \wedge \beta_{F(p)})(X_0, \dots, X_{k+\ell-1}) = (F^*\alpha_{F(p)} \wedge F^*\alpha_{F(p)})(X_0, \dots, X_{k+\ell-1}).$$

From the definition of pullback and wedge product we have

$$\begin{aligned} & F^*(\alpha_{F(p)} \wedge \beta_{F(p)})(X_0, \dots, X_{k+\ell-1}) \\ & = \alpha_{F(p)} \wedge \beta_{F(p)}\,(F_*X_0, \dots, F_*X_{k+\ell-1}) \\ & = \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}\sigma\, \alpha_{F(p)}(F_*X_{\sigma(0)}, \dots, F_*X_{\sigma(k-1)})\, \beta_{F(p)}(F_*X_{\sigma(k)}, \dots, F_*X_{\sigma(k+\ell-1)}) \\ & = \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}\sigma\, F^*\alpha_{F(p)}(X_{\sigma(0)}, \dots, X_{\sigma(k-1)})\, F^*\beta_{F(p)}(X_{\sigma(k)}, \dots, X_{\sigma(k+\ell-1)}) \\ & = (F^*\alpha_{F(p)} \wedge F^*\beta_{F(p)})(X_0, \dots, X_{k+\ell-1}). \end{aligned}$$

□

(b) In any coordinate chart y^i on N,

$$F^*\left(\sum_{\substack{I=(i_0,\dots,i_{k-1}) \\ 0 \le i_0,\dots,i_{k-1} < n}} \omega_I\,\mathrm{d}y^I \right) = \sum_{\substack{I=(i_0,\dots,i_{k-1}) \\ 0 \le i_0,\dots,i_{k-1} < n}} (\omega_I \circ F)\,\mathrm{d}(y^{i_0} \circ F) \wedge \cdots \wedge \mathrm{d}(y^{i_{k-1}} \circ F).$$

Solution. Observation: From the definition of the pullback F^* it follows immediately that

$$F^*(\omega + \eta) = F^*\omega + F^*\eta, \quad F^*(f\omega) = (f \circ F)F^*\omega$$

for $\omega, \eta \in \Omega^k(N)$, $f \in C^\infty(N)$. In addition to this we use the following properties of the pullback of 1-forms: let $f \in C^\infty(N)$, and $\omega \in \Omega^1(N)$, then $F^*\,\mathrm{d}f = \mathrm{d}(f \circ F)$ and $F^*(f\sigma) = (f \circ F)F^*\sigma$.

Denoting $I = (i_0, \ldots, i_{k-1})$ an increasing multi-index (i.e. such that $0 \leq i_0, \ldots, i_{k-1} < n$), we have

$$\begin{aligned} F^*\left(\sum_I \omega_I \, \mathrm{d}y^I\right) &= F^*\left(\sum_I \omega_I \, \mathrm{d}y^{i_0} \wedge \cdots \wedge \mathrm{d}y^{i_{k-1}}\right) \\ &= \sum_I (\omega_I \circ F) F^*(\mathrm{d}y^{i_0} \wedge \cdots \wedge \mathrm{d}y^{i_{k-1}}) \\ &= \sum_I (\omega_I \circ F)(F^* \, \mathrm{d}y^{i_0}) \wedge \cdots \wedge (F^* \, \mathrm{d}y^{i_{k-1}}) \\ &= \sum_I (\omega_I \circ F) \, \mathrm{d}(y^{i_0} \circ F) \wedge \cdots \wedge \mathrm{d}(y^{i_{k-1}} \circ F) \end{aligned}$$

□

(c) $F^*(\omega) \in \Omega^k(M)$.

Solution. The last line above is a local expression for $F^*\omega$ defined on the preimage of the domain of the chart (y^i) by F. The coefficients $\omega_I \circ F$ are smooth because $\omega_I = \omega_{i_0,\ldots,i_{k-1}}$ are the component functions of a k-form and hence smooth. The functions $y^{i_s} \circ F$ are smooth as well and hence their differentials are smooth 1-forms. Now the only missing ingredient is that the wedge product of a smooth differential forms is a smooth differential form. But this is clear because the component functions of the wedge product are sums of products of component functions of the original forms and hence smooth. □