Final Exam: 18 January 2017
Traitement Quantique de l'Information

Exercise 1 *Distribution imparfaite de clé secrète*

(a) Pour $a = 0$ ou 1 on a

$$\begin{aligned} P(x_i = y_i | e_i = d_i = a) = P(x_i = y_i = 0 | e_i = d_i = a, x_i = 0) P(x_i = 0) \\ + P(x_i = y_i = 1 | e_i = d_i = a, x_i = 1) P(x_i = 1) \end{aligned}$$

- Quand $a = 0$:

$$\begin{aligned} P(x_i = y_i = 0 | e_i = d_i = 0, x_i = 0) = |\langle \alpha | 0 \rangle|^2 = (\cos\alpha)^2 \\ P(x_i = y_i = 1 | e_i = d_i = 0, x_i = 1) = |\langle \alpha_\perp | 1 \rangle|^2 = (\cos\alpha)^2 \end{aligned}$$

$$P(x_i = y_i | e_i = d_i = 0) = \frac{1}{2}(\cos\alpha)^2 + \frac{1}{2}(\cos\alpha)^2 = (\cos\alpha)^2$$

- Quand $a = 1$:

$$\begin{aligned} P(x_i = y_i = 0 | e_i = d_i = 1, x_i = 0) = |\langle \alpha | H^\dagger H | 0 \rangle|^2 = (\cos\alpha)^2 \\ P(x_i = y_i = 1 | e_i = d_i = 1, x_i = 1) = |\langle \alpha_\perp | H^\dagger H | 1 \rangle|^2 = (\cos\alpha)^2 \end{aligned}$$

$$P(x_i = y_i | e_i = d_i = 1) = \frac{1}{2}(\cos\alpha)^2 + \frac{1}{2}(\cos\alpha)^2 = (\cos\alpha)^2$$

(b)

$$\begin{aligned} P(x_i = y_i | e_i = d_i) &= P(x_i = y_i | e_i = d_i, e_i = 0) P(e_i = 0) \\ &\quad + P(x_i = y_i | e_i = d_i, e_i = 1) P(e_i = 1) \\ &= \frac{1}{2}(\cos\alpha)^2 + \frac{1}{2}(\cos\alpha)^2 = (\cos\alpha)^2 \end{aligned}$$

Quand $\alpha = 0$ la probabilité est 1. Quand $\alpha = \frac{\pi}{2}$ la probabilité est 0. Quand $\alpha = \frac{\pi}{4}$ la probabilité est $\frac{1}{2}$.

Exercise 2 *Codage superdense imparfait*

(a) Supposons que l'on a le contraire

$$\begin{aligned} |\Psi\rangle &= \big(a|0\rangle + b|1\rangle\big) \otimes \big(c|0\rangle + d|1\rangle\big) \\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle . \end{aligned}$$

Alors $ac = ad = bd = \frac{1}{\sqrt{3}}$ et $bc = 0$. Cela implique que $b \neq 0$ car $bd \neq 0$ et $c = 0$ car $b \neq 0$ et $bc = 0$. Mais alors $ac = a(0) = 0 \neq \frac{1}{3}$ serait une contradiction. Donc $|\Psi\rangle$ est intriquée.

(b) Quand Bob recoit la photon d'Alice:

- Message 00 conduit à la paire $|\Psi\rangle = \frac{1}{\sqrt{3}}\big(\,|00\rangle + |01\rangle + |11\rangle\,\big)$
- Message 01 conduit à la paire $X\,|\Psi\rangle = \frac{1}{\sqrt{3}}\big(\,|10\rangle + |11\rangle + |01\rangle\,\big)$
- Message 10 conduit à la paire $Z\,|\Psi\rangle = \frac{1}{\sqrt{3}}\big(\,|00\rangle + |01\rangle - |11\rangle\,\big)$
- Message 11 conduit à la paire $iY\,|\Psi\rangle = \frac{1}{\sqrt{3}}\big(-|10\rangle - |11\rangle + |01\rangle\,\big)$

(c) Alice envoie 10. Bob possède donc la paire:

$$Z\,|\Psi\rangle = \frac{1}{\sqrt{3}}\big(\,|00\rangle + |01\rangle - |11\rangle\,\big).$$

Messages possibles observés par Bob:

- 00 avec la probabilité $|\,\langle B_{00}|\,Z\,|\Psi\rangle\,|^2 = \frac{1}{2}\cdot\frac{1}{3}(1-1)^2 = 0$
- 10 avec la probabilité $|\,\langle B_{10}|\,Z\,|\Psi\rangle\,|^2 = \frac{1}{6}(1)^2 = \frac{1}{6}$
- 01 avec la probabilité $|\,\langle B_{01}|\,Z\,|\Psi\rangle\,|^2 = \frac{1}{6}(1+1)^2 = \frac{2}{3}$
- 11 avec la probabilité $|\,\langle B_{11}|\,Z\,|\Psi\rangle\,|^2 = \frac{1}{6}(1)^2 = \frac{1}{6}$

Notez que la somme des probabilités est $0 + \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1$.

$$P(\text{erreur de transsmisson}) = P(\text{message} \neq 10) = P(00) + P(01) + P(11) = 0 + \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Exercise 3 *Dynamique du spin*

(a)

$$H = \hbar J \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hbar J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad e^{-\frac{t}{\hbar}H} = \begin{pmatrix} e^{-itJ} & 0 & 0 & 0 \\ 0 & e^{itJ} & 0 & 0 \\ 0 & 0 & e^{itJ} & 0 \\ 0 & 0 & 0 & e^{-itJ} \end{pmatrix}$$

(b)

$$\begin{aligned} R\left(\frac{\pi}{2}, \hat{y}\right)|\uparrow\rangle &= \cos\frac{\pi}{4}\,|\uparrow\rangle + i\sin\frac{\pi}{4}\sigma_y\,|\uparrow\rangle \\ &= \cos\frac{\pi}{4}\,|\uparrow\rangle + i\sin\frac{\pi}{4}(i\,|\downarrow\rangle) \\ &= \cos\frac{\pi}{4}\,|\uparrow\rangle - \sin\frac{\pi}{4}\,|\downarrow\rangle \\ &= \frac{1}{\sqrt{2}}\big(\,|\uparrow\rangle - |\downarrow\rangle\,\big) \end{aligned}$$

$$e^{-i\frac{t}{\hbar}H} I \otimes R\left(\frac{\pi}{2}, \hat{y}\right) |\uparrow\rangle \otimes |\uparrow\rangle = e^{-i\frac{t}{\hbar}H} |\uparrow\rangle \otimes \frac{1}{\sqrt{2}}\big(|\uparrow\rangle - |\downarrow\rangle \big)$$
$$= \frac{e^{-i\frac{t}{\hbar}H}}{\sqrt{2}}\big(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle \big)$$
$$= \frac{e^{-itJ}}{\sqrt{2}} |\uparrow\uparrow\rangle - \frac{e^{itJ}}{\sqrt{2}} |\uparrow\downarrow\rangle$$

$$R\left(\frac{\pi}{2}, \hat{x}\right) |\uparrow\rangle = \cos\frac{\pi}{4} |\uparrow\rangle + i\sin\frac{\pi}{4}\sigma_x |\uparrow\rangle = \cos\frac{\pi}{4} |\uparrow\rangle + i\sin\frac{\pi}{4} |\downarrow\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\rangle + i\,|\downarrow\rangle \big)$$
$$R\left(\frac{\pi}{2}, \hat{x}\right) |\downarrow\rangle = \frac{1}{\sqrt{2}}\big(|\downarrow\rangle + i\,|\uparrow\rangle \big)$$

Alors l'etat résultant est

$$U_t |\uparrow\rangle \otimes |\uparrow\rangle = R\left(\frac{\pi}{2}, \hat{x}\right) \otimes R\left(\frac{\pi}{2}, \hat{x}\right) e^{-i\frac{t}{\hbar}H} I \otimes R\left(\frac{\pi}{2}, \hat{y}\right) |\uparrow\rangle \otimes |\uparrow\rangle$$
$$= R\left(\frac{\pi}{2}, \hat{x}\right) \otimes R\left(\frac{\pi}{2}, \hat{x}\right) \left(\frac{e^{-itJ}}{\sqrt{2}} |\uparrow\uparrow\rangle - \frac{e^{itJ}}{\sqrt{2}} |\uparrow\downarrow\rangle\right)$$
$$= \frac{e^{-itJ}}{\sqrt{2}} \frac{|\uparrow\rangle + i\,|\downarrow\rangle}{\sqrt{2}} \otimes \frac{|\uparrow\rangle + i\,|\downarrow\rangle}{\sqrt{2}} + \frac{e^{itJ}}{\sqrt{2}} \frac{|\uparrow\rangle + i\,|\downarrow\rangle}{\sqrt{2}} \otimes \frac{|\downarrow\rangle + i\,|\uparrow\rangle}{\sqrt{2}}$$
$$= \frac{|\uparrow\rangle + i\,|\downarrow\rangle}{\sqrt{2}} \otimes \left\{ \frac{e^{-itJ} + ie^{itJ}}{2} |\uparrow\rangle + \frac{ie^{-itJ} + e^{itJ}}{2} |\downarrow\rangle \right\}$$

(c) Cet état est produit.