Introduction to Differentiable Manifolds

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Exercise 12.1. Show that the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is orientable.

Solution. Let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the projection map. For each open set $U \subseteq \mathbb{R}^n$ such that π is injective on U, the map $\varphi = \pi|_U$ is a local parametrization of $\mathbb{T}^n$, and the parametrizations of this kind form an atlas $\mathcal{A}$ of $\mathbb{T}^n$. Each transition map $\varphi^{-1} \circ \psi$, with $\varphi, \psi \in \mathcal{A}$, is locally a translation, therefore we have $\mathrm{D}_{\widetilde{p}}(\varphi^{-1} \circ \psi) = \mathrm{id}_{\mathbb{R}^n}$ for all points $\widetilde{p}$. Since $\operatorname{sgn}\det \mathrm{id}_{\mathbb{R}^n} = +1$, there is an orientation on $\mathbb{T}^n$ such that all the charts $\varphi \in \mathcal{A}$ are positive. □

Exercise 12.2. Show that a product $M_0 \times \cdots \times M_{k-1}$ of several orientable manifolds is naturally oriented.

Solution. Suppose each manifold M_i has an orientation $\mathcal{O}_i$. We will show how to produce a natural orientation, denoted $\prod_i \mathcal{O}_i$, on the product manifold $\prod_{0\le i<k} M_i$. Let $\pi_i : \prod_i M_i \to M_i$ be the projection map for $i \in \underline{k} = \{0, \dots, k-1\}$.

We first recall how we construct an atlas for the product manifold. Suppose we have for each manifold M_i a connected chart $\varphi_i : U_i \to \mathbb{R}^{n_i}$, where $n_i = \dim M_i$. (A "connected chart" is a chart whose domain is connected. Note that any manifold is covered by its connected charts. Here we require that each chart φ_i be connected to ensure that it has a constant sign $\operatorname{sgn}_{\mathcal{O}_i} \varphi_i = \pm 1$ according to the orientatation $\mathcal{O}_i$.) Using the charts φ_i we construct a product chart

$$\begin{array}{rccc} \Phi = \prod_i \varphi_i \circ \pi_i : & \prod_i U_i & \to & \prod_i \mathbb{R}^{n_i} \\ & x & \mapsto & (\varphi_i(\pi_i(x)))_i \end{array}$$

of the product manifold $\prod_i M_i$. The charts $\Phi = \prod_i \varphi_i \circ \pi_i$ of this kind form an atlas $\mathcal{A}$ of the manifold $\prod_i M_i$. Let us define the sign of a chart $\Phi = \prod_i \varphi_i \circ \pi_i$ as above as

$$s(\Phi) := \prod_i \operatorname{sgn}_{\mathcal{O}_i} \varphi_i.$$

Let us show that these function $s : \mathcal{A} \to \{\pm 1\}$ is an orientation sign function. Suppose we have a second chart $\Psi = \prod_i \psi_i \circ \pi_i$, where each (V_i, ψ_i) is a chart of M_i. Then the differential of the transition map $\Psi \circ \Phi^{-1}$ at any point $\widetilde{P} = \prod_i \varphi_i(p_i)$, where $p_i \in U_i \cap V_i$, is represented by a diagonal block matrix

$$\begin{aligned} \left[\, \mathrm{D}_P \left(\Phi \circ \Phi^{-1}\right) \right] &= \operatorname{diag} \left[\, \mathrm{D}_{\varphi(p_i)}(\psi_i \circ \varphi_i^{-1}) \right]_i \\ &= \begin{pmatrix} \left[\, \mathrm{D}_{\varphi(p_0)}(\psi_0 \circ \varphi_0^{-1}) \right] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left[\, \mathrm{D}_{\varphi(p_{k-1})}(\psi_{k-1} \circ \varphi_{k-1}^{-1}) \right] \end{pmatrix}. \end{aligned}$$

The determinant of this matrix is

$$\det \left[\, \mathrm{D}_P \left(\Phi \circ \Phi^{-1}\right) \right] = \prod_i \det \left[\, \mathrm{D}_{\varphi(p_i)}(\psi_i \circ \varphi_i^{-1}) \right],$$

and the sign of this determinant is

$$\begin{aligned} \operatorname{sgn}\det \left[\, \mathrm{D}_P \left(\Phi \circ \Phi^{-1}\right) \right] = \prod_i \operatorname{sgn}\det \left[\, \mathrm{D}_{\varphi(p_i)}(\psi_i \circ \varphi_i^{-1}) \right] &= \prod_i \operatorname{sgn} \psi_i \cdot \operatorname{sgn} \varphi_i \\ &= s(\Phi) \cdot s(\Psi). \end{aligned}$$

(Here we used the fact that $\operatorname{sgn}\det \left[\, \mathrm{D}_{\varphi(p_i)}(\psi_i \circ \varphi_i^{-1}) \right] = \operatorname{sgn}_{\mathcal{O}_i} \psi_i \cdot \operatorname{sgn}_{\mathcal{O}_i} \varphi_i$.) Thus there is an orientation, denoted $\prod_i \mathcal{O}_i$, on the product manifold $\prod_i M_i$ such that $\operatorname{sgn}_{\prod_i \mathcal{O}_i}(\Phi) = s(\Phi)$ for each connected chart $\Phi = \prod_i \varphi_i \circ \pi_i$ as constructed above. □

Exercise 12.3. Let M be a differentiable n-manifold. Show that the sign $\operatorname{sgn}\omega$ of a nonvanishing n-form on M, defined by

$$(\operatorname{sgn}\omega)|_p(X_0,\dots,X_{n-1}) := \operatorname{sgn}(\omega|_p(X_0,\dots,X_{n-1}))$$

for $p \in M$ and $X_0,\dots,X_{n-1} \in \mathrm{T}_pM$, is an orientation on M.

Solution. First part: Let us check that for each point $p \in M$, the function $\mathcal{O}_p = \operatorname{sgn}\circ\omega_p$ that sends a base $B = (B_i)_{0\le i<n}$ of T_pM to the number $\operatorname{sgn}(\omega|_p(B_0,\dots,B_{n-1})) = \pm 1$ is an orientation on T_pM. We have to show that

$$\mathcal{O}_p(BC) = \mathcal{O}_p(B)\cdot \operatorname{sgn}\det C,$$

for any base B and any invertible matrix $C \in \mathbb{R}^{n\times n}$. For this it suffices to show that

$$\omega_p(BC) = \omega_p(B)\cdot\det C$$

for any base B and any invertible matrix $C \in \mathbb{R}^{n\times n}$. Since any invertible matrix C can be decomposed as a product of elementary matrices (such as $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $\begin{pmatrix}1&0&0\\0&1&\lambda\\0&0&1\end{pmatrix}$, $\begin{pmatrix}1&0&0\\0&\lambda&0\\0&0&1\end{pmatrix}$) , it is sufficient to prove that $\omega_p(BC) = \omega_p(B)\cdot\det C$ when C is an elementary matrix. And this, in turn, folows from the fact that ω_p is multilinear and alternating:

- If C is a transposition matrix (which coincides with the identity matrix except at four coefficients $T_{i,i} = T_{j,j} = 0$, $T_{i,j} = T_{j,i} = 1$), then $\omega_p(BC) = -\omega_p(B)$ since ω_p is skew-symmetric. On the other hand we have $\det C = -1$, thus the equation $\omega_p(BC) = \omega_p(B)\cdot\det C$ holds.
- If $C = I + \lambda E_{j,i}$, with $i \neq j$, then

  $$\omega_p(BC) = \omega_p(B_0,\dots,B_i+\lambda B_j,\dots,B_{n-1}) = \omega_p(B)$$

  since ω_p is multilinear and alternating. Since on the other hand we have $\det C = 1$, the equation $\omega_p(BC) = \omega_p(B)\cdot\det C$ holds.
- If C is a diagonal matrix where $C_{i,i} = \lambda \in \mathbb{R}$ and $C_{j,j} = 1$ for $j \neq i$, then

  $$\omega_p(BC) = \omega_p(B_0,\dots,\lambda B_i,\dots,B_{n-1}) = \lambda\omega_p(B_0,\dots,B_{n-1})$$

  by linearity of ω_p. On the other hand we have $\det C = \lambda$, thus again the equation $\omega_p(BC) = \omega_p(B)\cdot\det C$ holds.

Second part: Let us show that the orientation field $\mathcal{O}$ is continuous. We take a connected chart (U,φ) and write $\omega|_U = h\,\mathrm{d}\varphi^0\wedge\cdots\wedge\mathrm{d}\varphi^{n-1}$, where $h : U \to \mathbb{R}$ is a continuous function. Note that h does not vanish, therefore it has constant sign. Therefore the map

$$p\in U \mapsto \mathcal{O}_p\left(\left.\frac{\partial}{\partial\varphi^0}\right|_p,\dots,\left.\frac{\partial}{\partial\varphi^{n-1}}\right|_p\right) = \operatorname{sgn}\omega_p\left(\left.\frac{\partial}{\partial\varphi^0}\right|_p,\dots,\left.\frac{\partial}{\partial\varphi^{n-1}}\right|_p\right) = \operatorname{sgn}h(p)$$

is constant on U. This shows that the orientation $\mathcal{O}$ is continuous on U. □

Exercise 12.4. Show that the projective plane $\mathbb{P}^2$ is not orientable.

Solution. Recall that the projective plane is $\mathbb{P}^2 := \mathbb{R}^3\setminus\{0\}/\sim$, where $x\sim y$ iff $x = \lambda y$ for $\lambda\in\mathbb{R}\setminus\{0\}$. Assume, for a contradiction, that $\mathbb{P}^2$ is orientable.

We will use the charts (U,φ), (V,ψ), where

$$U = \{[x,y,z]\in\mathbb{P}^2 : z\neq 0, \qquad \varphi([x,y,z]) = \left(\frac{x}{z},\frac{y}{z}\right)$$
$$V = \{[x,y,z]\in\mathbb{P}^2 : x\neq 0, \qquad \psi([x,y,z]) = \left(\frac{y}{x},\frac{z}{x}\right).$$

These two charts do not cover the manifold $\mathbb{P}^2$, but they are enough for our purpose. Since these charts are connected, to prove that $\mathbb{P}^2$ is not orientable it suffices to show that the determinant of the differential of transition map $\varphi \circ \psi^{-1}$ does not have constant sign.

Let us compute this determinant. Since $\psi^{-1}(y, z) = [1, y, z]$, the transition map is

$$\varphi \circ \psi^{-1} : (y, z) \mapsto [1, y, z] \mapsto \left(\frac{1}{z}, \frac{y}{z}\right),$$

defined for $(y, z) \in \mathbb{R}^2$ such that $z \neq 0$. The differential of this transition map is represented by the matrix

$$[\mathrm{D}_{(y,z)}(\varphi \circ \psi^{-1})] = \begin{pmatrix} 0 & -\frac{1}{z^2} \\ \frac{1}{z} & -\frac{y}{z^2} \end{pmatrix}$$

which has determinant

$$\det[\mathrm{D}_{(y,z)}(\varphi \circ \psi^{-1})] = \det \begin{pmatrix} 0 & -\frac{1}{z^2} \\ \frac{1}{z} & -\frac{y}{z^2} \end{pmatrix} = \frac{1}{z^3}.$$

This value does not have constant sign, as we had to show.

□

Exercise 12.5. Show that the Möbius band is not orientable.

Solution. The Möbius band M is the quotient space of $\mathbb{R}^2$ by the equivalence relation

$$(x, y) \sim (x', y') \iff x' - x = n \in \mathbb{Z} \text{ and } y' = (-1)^n y$$

Let us show that it is not orientable. Suppose, for a contradiction, that M has an orientation $\mathcal{O}$. Let $\pi : \mathbb{R}^2 \to M$ be the quotient map. We take an atlas consisting of two parametrizations $\varphi = \pi|_U$ and $\psi = \pi|_V$, where $U = (0, 1) \times \mathbb{R}$ and $V = (\frac{1}{2}, \frac{3}{2}) \times \mathbb{R}$. Since the two sets U, V are connected, each of the two parametrizations φ, ψ has a constant sign w.r.t. the orientation $\mathcal{O}$, therefore the transition map $\psi^{-1} \circ \varphi$ must have a constant sign as well. However, the transition map is

$$\psi^{-1} \circ \varphi(x, y) = \begin{cases} (x, y) & \text{if } x \in (\frac{1}{2}, 1) \\ (x, -y) & \text{if } x \in (0, \frac{1}{2}) \end{cases}$$

and the sign

$$\operatorname{sgn} \det \mathrm{D}_{(x,y)}(\psi^{-1} \circ \varphi) = \begin{cases} 1 & \text{if } x \in (\frac{1}{2}, 1) \\ -1 & \text{if } x \in (0, \frac{1}{2}) \end{cases}$$

is not constant. □

1. Manifolds with boundary

Exercise 12.6. Let M be an n-dimensional $\mathcal{C}^r$ manifold with boundary.

(a) Show that its interior $\operatorname{Int} M$ and its boundary ∂M are disjoint subsets.

Solution. Suppose there is point p contained on both ∂M and $\operatorname{Int} M$, therefore there are two charts $\varphi : U \to \widetilde{U}$ and $\psi : V \to \widetilde{V}$, where $U, V \subseteq M$ and $\widetilde{U}, \widetilde{V} \subseteq \mathbb{H}^n$ are open sets, such that $\varphi(p) \in \operatorname{Int} \mathbb{H}^n$ and $\psi(p) \in \partial \mathbb{H}^n$.

We may assume (by shrinking $\widetilde{U}$ to $\widetilde{U} \cup \operatorname{Int} \mathbb{H}^n$ and shhrinking U accordingly to $\varphi^{-1}(\widetilde{U} \cup \operatorname{Int} \mathbb{H}^n)$ and restricting φ) that the set $\widetilde{U}$ is contained in $\operatorname{Int} \mathbb{H}^n$ and therefore it is an open subset of $\mathbb{R}^n$. Also, we may assume that $U = V$ (again, by shrinking U and V to $U \cap V$, and shrinking $\widetilde{U}$ and $\widetilde{V}$ accordingly to $\varphi(U \cap V$ and $\psi(U \cap V)$ resp.). Therefore the transition map $\psi \circ \varphi^{-1} : \widetilde{U} \to \widetilde{V}$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ and $\mathbb{H}^n$respectively. In particular, the transition map $\psi \circ \varphi^{-1}$ is $\mathcal{C}^1$ and has injective differential at every point. This implies, by the inverse function theorem, that its image $\widetilde{V}$ is an open subset of $\mathbb{R}^n$. But since $\widetilde{V} \subseteq \mathbb{H}^n$, this means that $\widetilde{V} \subseteq \operatorname{Int}\mathbb{H}^n$. This contradicts the fact that $\psi(p) \in \partial\mathbb{H}^n$.

Let us show that $\operatorname{Int} M$ is an open subset of M. Since

$$\operatorname{Int}(M) = \bigcup_{(U,\varphi)\text{ chart}} U \cap \operatorname{Int} M,$$

thus it suffices to show that $U \cap \operatorname{Int} M$ is open in U for all charts (U, φ). And indeed, we have $U \cap \operatorname{Int} M = \varphi^{-1}(\operatorname{Int}\mathbb{H}^n)$, and this is an open subset of U since φ is continuous and $\operatorname{Int}\mathbb{H}^n$ is an open subset of $\mathbb{H}^n$. □

(b) Show that ∂M, endowed with the subspace topology, can be given the structure of an $n-1$-dimensional $\mathcal{C}^r$ manifold (without boundary) such that the inclusion map into M is $\mathcal{C}^r$.

Solution. This is done in the lectures. □

Exercise 12.7. Prove that a continuous k-form is determined by the value of its integrals (Proposition 7.3.12). ***Hint:*** Use a chart to move the problem to $\mathbb{R}^n$, then integrate on small pieces of coordinate planes.

Solution. Suppose that our manifold is an open set $U \subseteq \mathbb{R}^n$ (considered as a $\mathcal{C}^1$ manifold) and $\omega \in \Omega^k(U)$ is a continuous k-form. We write $\omega = \sum_{I \in \underline{n}^k_{\nearrow}} \omega_I \,\mathrm{d}x^I$. Let $p \in U$ and $I = (i_0, \dots, i_{k-1}) \in \underline{n}^k$ an increasing k-index. We want to show that $\omega_I(p)$ is determined by values of k-dimensional integrals of ω.

Let $\iota = \iota_{p,I} : \mathbb{R}^k \to \mathbb{R}^n : y \mapsto x$ where

$$x^i = \begin{cases} y^s & \text{if } i = i_s \text{ for some } s \in \underline{k} \\ p^i & \text{otherwise.} \end{cases}$$

We restrict ι to the open set $V = \iota^{-1}U$ and note that the point $\widetilde{p} = \iota^{-1}(p) = (p^{i_s})_{s\in\underline{k}} \subseteq \mathbb{R}^k$ is contained in V.

The pullback by ι of ω is the continuous k-form $\iota^*\omega = h\,\mathrm{d}y^0 \wedge \cdots \wedge \mathrm{d}y^{k-1} \in \Omega^k(V)$, where $h = \omega_I \circ \iota \in \mathcal{C}(V, \mathbb{R})$.

Denote $D_{\widetilde{p},\varepsilon} \subseteq \mathbb{R}^k$ be the closed ball of center $\widetilde{p}$ and radius $\varepsilon > 0$ in $\mathbb{R}^k$, and let $|D_{\widetilde{p},\varepsilon}|$ be the volume of this ball. We take ε small enough so that $D_{\widetilde{p},\varepsilon} \subseteq V$. Then

$$\frac{1}{|D_{\widetilde{p},\varepsilon}|}\int_{D_{\widetilde{p},\varepsilon}} \iota^*\omega = \frac{1}{|D_{\widetilde{p},\varepsilon}|}\int_{D_{\widetilde{p},\varepsilon}} h = (\text{average value of } h \text{ on } D_{\widetilde{p},\varepsilon}) \xrightarrow{\varepsilon\to 0} h(\widetilde{p}) = \omega_I(p).$$

This means that we can find out the value of $\omega_I(p)$ if we know the value of the integral $\int_{D_{\widetilde{p},\varepsilon}} \iota^*_{p,I}\omega$ for every $\varepsilon > 0$. Doing this for each increasing k-index $I \in \underline{n}^k_{\nearrow}$, we find out the value of $\omega|_p$ at the point $p \in U$. Thus any continuous k-form $\omega \in \Omega^k(U)$ is determined by the value of its integrals of the form $\int_{D_{\widetilde{p},\varepsilon}} \iota^*_{p,\varepsilon}\omega$.

If M is a general $\mathcal{C}^1$ manifold and $\omega \in \Omega^k(M)$ is a continuous k-form, we use a local parametrization $\varphi : U \to M$ to get a k-form $\varphi^*\omega \in \Omega^k(U)$, and then as explained above we can determine this k-form if we know the value of the integrals of the kind

$$\int_{D_{\widetilde{p},\varepsilon}} \iota^*_{p,I}(\varphi^*\omega) = \int_{D_{\widetilde{p},\varepsilon}} (\varphi \circ \iota_{p,I})^*\omega$$

But knowing $\varphi^*\omega$ is equivalent to knowing $\omega|_{\varphi(U)}$, thus using different parametrizations (U, φ) we can know ω at all points of M. □