Introduction to Differentiable Manifolds
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Solutions Series 13 - Exterior derivative, Stokes' theorem 2022–01–02

Exercise 13.1 (Properties of the integral)**.** Let M be an oriented differentiable n-manifold and let ω, η be two continuous, compactly supported n-forms on M. Prove the following:

(a) Linearity: If $a, b \in \mathbb{R}$, then

$$\int_M (a\,\omega + b\,\eta) = a\int_M \omega + b\int_M \eta.$$

Solution. First case: The manifold M is an open subset U of $\mathbb{R}^n$, with the standard orientation. Then we may write

$$\omega = h\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$$
$$\eta = g\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$$

and we have $a\,\omega + b\,\eta = (a\,h + b\,g)\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$, therefore

$$\int_M (a\,\omega + b\,\eta) = \int_U (a\,h + b\,g) = a\int_U h + b\int_M g = a\int_M \omega + b\int_M \eta$$

where the integrals over U are Riemann integrals.

Second case: M is a general manifold, but the supports of ω and η are contained in the image $\varphi(U)$ of a single parametrization $\varphi : U \to M$ that has constant sign $\operatorname{sgn}\varphi = \pm 1$. Then $\operatorname{supp}(a\,\omega + b\,\eta) \subseteq \operatorname{supp}\omega \cup \operatorname{supp}\eta \subseteq \varphi(U)$, and we have

$$\begin{aligned}\int_M (a\,\omega + b\,\eta) &= \operatorname{sgn}\varphi \cdot \int_U \varphi^*(a\,\omega + b\,\eta)\\ &= \operatorname{sgn}\varphi \cdot \int_U (a\,\varphi^*\omega + b\,\varphi^*\eta)\\ &= a\operatorname{sgn}\varphi \cdot \int_U \varphi^*\omega + b\operatorname{sgn}\varphi \cdot \int_U \varphi^*\eta = a\int_M \omega + b\int_M \eta\end{aligned}$$

General case: We use a family of local parametrizations (U_i, φ_i) whose images $\varphi_i(U_i)$ cover $\operatorname{supp}\omega$ and $\operatorname{supp}\eta$, and a partition of unity $(\chi_i)_i$ subordinate to the open cover $(U_i)_i$ of the set $\bigcup_i \varphi_i(U_i)$. Then by definition of $\int_M$ we have

$$\begin{aligned}\int_M (a\,\omega + b\,\eta) &= \sum_i \int_M \chi_i(a\,\omega + b\,\eta)\\ &= \sum_i \int_M (a\,\chi_i\,\omega + b\,\chi_i\,\eta).\end{aligned}$$

Since the forms $\chi_i\,\omega$ and $\chi_i\,\eta$ have their support contained in $\varphi_i(U_i)$, by the previous case the last integral is equal to

$$\begin{aligned}&= \sum_i \left(a\int_M \chi_i\,\omega + b\int_M \chi_i\,\eta\right)\\ &= a\sum_i \int_M \chi_i\,\omega + b\sum_i \int_M \chi_i\,\eta\\ &= a\int_M \omega + b\int \eta.\end{aligned}$$

□

(b) Positivity: If $\operatorname{sgn}(\omega|_p)$ coincides with the orientation of M at every point $p \in M$ where $\omega|_p \neq 0$, then $\int_M \omega \geq 0$, and the inequality is strict unless ω is identically zero.

Solution. First case: Let M be an open set $U \subseteq \mathbb{R}^n$, with the standard orientation. In this case we can write $\omega = h \, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$ for some continuous function $h : U \to \mathbb{R}$, and we have $\operatorname{sgn}(\omega_p) = \operatorname{sgn} h(p)$ for any point $p \in U$, because $\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$ is a positive n-form. Therefore the condition that $\operatorname{sgn}(\omega_p)$ coincides with the orientation of M for all points $p \in U$ where $\omega_p \neq 0$ is equivalent to the condition that $h(p) \geq 0$ for all $p \in U$.

Thus we have $\int_M \omega = \int_U h \geq 0$, and the inequality is strict unless $h \equiv 0$, which means $\omega \equiv 0$.

Second case: Suppose $\operatorname{supp} \omega \subseteq U$ for some local parametrization $\varphi : \widetilde{U} \to U$ that has constant sign $\operatorname{sgn} \varphi = \pm 1$. For a point $p = \varphi(\widetilde{p})$ where $\omega_p \neq 0$, we have

$$\operatorname{sgn}((\varphi^* \omega)|_p) = \operatorname{sgn} \varphi \cdot \underbrace{\operatorname{sgn}(\omega|_p)}_{=+1},$$

which implies that $\operatorname{sgn}(\varphi^* \omega)|_p = \operatorname{sgn} \varphi$. Therefore

$$\int_M \omega = \operatorname{sgn} \varphi \int_{\widetilde{U}} \varphi^* \omega \geq 0$$

and the equality holds if and only if $\varphi^* \omega \equiv 0$, iff $\omega \equiv 0$.

General case: We use a family of local parametrizations (U_i, φ_i) whose images $\varphi_i(U_i)$ cover $\operatorname{supp} \omega$ and $\operatorname{supp} \eta$, and a partition of unity $(\chi_i)_i$ subordinate to the open cover $(U_i)_i$ of the set $\bigcup_i \varphi_i(U_i)$. Since $\chi_i \geq 0$, we have $\operatorname{sgn}(\chi_i \omega) \geq 0$, and therefore $\int_M \omega = \sum_i \int_M \chi_i \omega \geq 0$ with equality iff $\chi_i \omega = 0$ for all i, iff $\omega \equiv 0$. ☐

(c) Diffeomorphism invariance: If $f : N \to M$ is an diffeomorphism of constant sign $\operatorname{sgn}(f) = \pm 1$ (i.e. f is either orientation preserving or orientation reversing), then

$$\int_N f^* \omega = \operatorname{sgn} f \cdot \int_M \omega.$$

Solution. Suppose $\operatorname{supp}(f^* \omega) \subseteq \varphi(\widetilde{U})$ for some local parametrization $\varphi : \widetilde{U} \to N$. Then $f \circ \varphi$ is a local parametrization of M such that $\operatorname{supp} \omega \subseteq f(\varphi(U))$, and we have

$$\int_N f^* \omega = \int_{\widetilde{U}} \varphi^* (f^* \omega) = \int_{\widetilde{U}} (f \circ \varphi)^* \omega = \int_M \omega.$$

In the general case, the result is deduced easily using partitions unity. ☐

(d) Orientation reversal: If $-M$ denotes M with the reversed orientation, then

$$\int_{-M} \omega = - \int_M \omega.$$

Solution. Let $\mathcal{O}$ be the orientation of M, so that $-\mathcal{O}$ is the orientation of $-M$. Here the most important case is when $\operatorname{supp}(\omega) \subseteq \varphi(\widetilde{U})$ for some local parametrization $\varphi : \widetilde{U} \to M$. In the formula

$$\int_M \omega := \operatorname{sgn}_{\mathcal{O}} \varphi \cdot \int_{\widetilde{U}} \varphi^* \omega,$$

if we reverse the orientation of M, then the sign of φ is also reversed:

$$\int_{-M} \omega = \operatorname{sgn}_{-\mathcal{O}} \varphi \cdot \int_{\widetilde{U}} \varphi^* \omega = - \operatorname{sgn}_{\mathcal{O}} \varphi \cdot \int_{\widetilde{U}} \varphi^* \omega = - \int_M \omega.$$

In the general case, the result is deduced easily using partitions unity. ☐

Exercise 13.2. Prove that a continuous k-form is determined by the value of its integrals (Proposition 7.3.12). ***Hint:*** Use a chart to move the problem to $\mathbb{R}^n$, then integrate on small pieces of coordinate planes.

Solution. Suppose that our manifold is an open set $U \subseteq \mathbb{R}^n$ (considered as a $\mathcal{C}^1$ manifold) and $\omega \in \Omega^k(U)$ is a continuous k-form. We write $\omega = \sum_{I \in \underline{n}^k_{\nearrow}} \omega_I \,\mathrm{d}x^I$. Let $p \in U$ and $I = (i_0, \dots, i_{k-1}) \in \underline{n}^k$ an increasing k-index. We want to show that $\omega_I(p)$ is determined by values of k-dimensional integrals of ω.

Let $\iota = \iota_{p,I} : \mathbb{R}^k \to \mathbb{R}^n : y \mapsto x$ where

$$x^i = \begin{cases} y^s & \text{if } i = i_s \text{ for some } s \in \underline{k} \\ p^i & \text{otherwise.} \end{cases}$$

We restrict ι to the open set $V = \iota^{-1}U$ and note that the point $\widetilde{p} = \iota^{-1}(p) = (p^{i_s})_{s \in \underline{k}} \subseteq \mathbb{R}^k$ is contained in V.

The pullback by ι of ω is the continuous k-form $\iota^*\omega = h \,\mathrm{d}y^0 \wedge \dots \wedge \mathrm{d}y^{k-1} \in \Omega^k(V)$, where $h = \omega_I \circ \iota \in \mathcal{C}(V, \mathbb{R})$.

Denote $D_{\widetilde{p},\varepsilon} \subseteq \mathbb{R}^k$ be the closed ball of center $\widetilde{p}$ and radius $\varepsilon > 0$ in $\mathbb{R}^k$, and let $|D_{\widetilde{p},\varepsilon}|$ be the volume of this ball. We take ε small enough so that $D_{\widetilde{p},\varepsilon} \subseteq V$. Then

$$\frac{1}{|D_{\widetilde{p},\varepsilon}|} \int_{D_{\widetilde{p},\varepsilon}} \iota^*\omega = \frac{1}{|D_{\widetilde{p},\varepsilon}|} \int_{D_{\widetilde{p},\varepsilon}} h = (\text{average value of } h \text{ on } D_{\widetilde{p},\varepsilon}) \overset{\varepsilon \to 0}{\longrightarrow} h(\widetilde{p}) = \omega_I(p).$$

This means that we can find out the value of $\omega_I(p)$ if we know the value of the integral $\int_{D_{\widetilde{p},\varepsilon}} \iota^*_{p,I}\omega$ for every $\varepsilon > 0$. Doing this for each increasing k-index $I \in \underline{n}^k_{\nearrow}$, we find out the value of $\omega|_p$ at the point $p \in U$. Thus any continuous k-form $\omega \in \Omega^k(U)$ is determined by the value of its integrals of the form $\int_{D_{\widetilde{p},\varepsilon}} \iota^*_{p,\varepsilon}\omega$.

If M is a general $\mathcal{C}^1$ manifold and $\omega \in \Omega^k(M)$ is a continuous k-form, we use a local parametrization $\varphi : U \to M$ to get a k-form $\varphi^*\omega \in \Omega^k(U)$, and then as explained above we can determine this k-form if we know the value of the integrals of the kind

$$\int_{D_{\widetilde{p},\varepsilon}} \iota^*_{p,I}(\varphi^*\omega) = \int_{D_{\widetilde{p},\varepsilon}} (\varphi \circ \iota_{p,I})^*\omega$$

But knowing $\varphi^*\omega$ is equivalent to knowing $\omega|_{\varphi(U)}$, thus using different parametrizations (U, φ) we can know ω at all points of M. □

Exercise 13.3.* Let $f : M \to N$ be a smooth map between smooth manifolds. Then for all $\omega \in \Omega^k(M)$ we have

$$f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega).$$

Exercise 13.4.* Let (x, y, z) be the standard coordinates on $\mathbb{R}^3$ and let (v, w) be the standard coordinates on $\mathbb{R}^2$. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as $\phi(x, y, z) = (x + z, xy)$. Let $\alpha = e^w \,\mathrm{d}v + v \,\mathrm{d}w$ and $\beta = v \;\, \mathrm{d}v \wedge \mathrm{d}w$ be 2-forms on $\mathbb{R}^2$. Compute the following differential forms:

$$\alpha \wedge \beta, \quad \phi^*(\alpha), \quad \phi^*(\beta), \quad \phi^*(\alpha) \wedge \phi^*(\beta).$$

Solution. (a) Since $\Omega^3(\mathbb{R}^2)$ is just the zero form, whatever α and β are, $\alpha \wedge \beta = 0$

(b) We can either use the definition of $\phi^*(\alpha)$ to compute the components of this 1-form, or the property that, in local coordinates,

$$\phi^*(\alpha_I \,\mathrm{d}y^I) = (\alpha_I \circ \phi) \,\mathrm{d}(y^I \circ \phi)$$

If we want to use the definition, consider that a basis for $\Omega^1(\mathbb{R}^3)$ is given by $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$. Following a standard convention, we denote for convenience $\frac{\partial}{\partial x} = \partial_x$. Thus we have

$$\begin{aligned} \phi^*(\alpha)(\partial_x) &= \alpha(\mathrm{d}\phi(\partial_x)) = \alpha(\partial_v + y\partial_w) = e^w + yv = e^{xy} + (x+z)y \\ \phi^*(\alpha)(\partial_y) &= \alpha(\mathrm{d}\phi(\partial_y)) = \alpha(x\partial_w) = xv = (x+z)x \\ \phi^*(\alpha)(\partial_z) &= \alpha(\mathrm{d}\phi(\partial_z)) = \alpha(\partial_v) = e^w = e^{xy} \end{aligned}$$

Thus we obtain

$$\phi^*\alpha = e^{xy} + (x+z)y \,\mathrm{d}x + (x+z)x \,\mathrm{d}y + e^{xy} \,\mathrm{d}z$$

(c) To compute $\phi^*\beta$ we use the other (perhaps more direct) method. It uses the following important property of the pull-back:

$$\phi^*(f(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k) = (f \circ \phi)\,\mathrm{d}(x^1 \circ \phi) \wedge \cdots \wedge \mathrm{d}(x^k \circ \phi)$$

Thus we have

$$\begin{aligned}\phi^*(\beta) &= \phi^*(v\ \ \mathrm{d}v \wedge \mathrm{d}w)\\ &= v(\phi)\,\mathrm{d}(v(\phi)) \wedge \mathrm{d}(w(\phi))\\ &= (x+z)(\mathrm{d}x + \mathrm{d}z) \wedge (y\,\mathrm{d}x + x\,\mathrm{d}y)\\ &= (x+z)(x\,\mathrm{d}x \wedge \mathrm{d}y - y\,\mathrm{d}x \wedge \mathrm{d}z - x\,\mathrm{d}y \wedge \mathrm{d}z)\end{aligned}$$

(d) The property $\phi^*(\alpha) \wedge \phi^*(\beta) = \phi^*(\alpha \wedge \beta)$ implies that this is a null form.

□

Exercise 13.5.* Compute the exterior derivative of the following forms:

(a) on $\mathbb{R}^2 \setminus \{0\}$ $\theta = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}$.

Solution. Applying the definition of exterior derivative yields

$$\mathrm{d}\theta = \mathrm{d}\left(\frac{x}{x^2+y^2}\right)\wedge \mathrm{d}y + \mathrm{d}\left(\frac{-y}{x^2+y^2}\right)\wedge \mathrm{d}x = \frac{-x^2+y^2}{(x^2+y^2)^2}\,\mathrm{d}x\wedge\mathrm{d}y - \frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}y\wedge\mathrm{d}x = 0$$

□

(b) on $\mathbb{R}^3$, $\varphi = \cos(x)\,\mathrm{d}y \wedge \mathrm{d}z$.

Solution. $\mathrm{d}\varphi = \mathrm{d}(\cos(x)) \wedge \mathrm{d}y \wedge \mathrm{d}z = -\sin(x)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. □

(c) on $\mathbb{R}^3$ $\omega = A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z$.

Solution. Here consider A, B, C as smooth functions on $\mathbb{R}^3$, by definition we have

$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}A \wedge \mathrm{d}x + \mathrm{d}B \wedge \mathrm{d}y + \mathrm{d}C \wedge \mathrm{d}z\\ &= \left(\frac{\partial A}{\partial x}\mathrm{d}x + \frac{\partial A}{\partial y}\mathrm{d}y + \frac{\partial A}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x + \left(\frac{\partial B}{\partial x}\mathrm{d}x + \frac{\partial B}{\partial y}\mathrm{d}y + \frac{\partial B}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y\\ &+ \left(\frac{\partial C}{\partial x}\mathrm{d}x + \frac{\partial C}{\partial y}\mathrm{d}y + \frac{\partial C}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z\\ &= -\frac{\partial A}{\partial y}\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial A}{\partial z}\mathrm{d}z \wedge \mathrm{d}x + \frac{\partial B}{\partial x}\mathrm{d}x \wedge \mathrm{d}y - \frac{\partial B}{\partial z}\mathrm{d}y \wedge \mathrm{d}z - \frac{\partial C}{\partial x}\mathrm{d}z \wedge \mathrm{d}x + \frac{\partial C}{\partial y}\mathrm{d}y \wedge \mathrm{d}z\\ &= \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y - \left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x\end{aligned}$$

Remark: Notice the resemblance of $\mathrm{d}\omega$ with the curl of a vector field in $\mathbb{R}^3$. Recall that if $F = (A, B, C)$ is a vector field in $\mathbb{R}^3$ then

$$\mathrm{curl}F = (C_y - B_z, A_z - C_x, B_x - A_y)$$

One can identify 2-forms with vector fields, by sending $a\,\mathrm{d}x \wedge \mathrm{d}y + b\,\mathrm{d}y \wedge \mathrm{d}z + c\,\mathrm{d}z \wedge \mathrm{d}x$ to the vector field (b, c, a). □

Exercise 13.6.* Deduce the following classical theorems from Stokes' theorem.

(a) **Green's theorem.** Let $D \subseteq \mathbb{R}^2$ be a smooth 2-dimensional compact embedded submanifold with boundary in $\mathbb{R}^2$. Then for any differentiable 1-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$ defined on an open neighborhood of D we have

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\ \mathrm{d}y = \int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

Solution. The differential of ω is

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}P \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y \\ &= \left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}y \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Hence this is a particular case of the Stokes' theorem. □

(b) **Divergence theorem.** Let $A \subset \mathbb{R}^3$ be a 3-dimensional compact embedded submanifold with boundary in $\mathbb{R}^3$. Then for any smooth vector field $F : A \to \mathbb{R}^3$ we have

$$\int_A \mathrm{div}F \;\mathrm{d}V = \int_{\partial A} F \cdot \mathrm{d}$$

wher $\mathrm{d}V := \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is the standard 3-form on $\mathbb{R}^3$ and on the right hand side we have the formal inner product with $\mathrm{d}S = (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$.

Solution. The expression $F \cdot \mathrm{d}S$ is to be understood as in the lecture notes as the 2-form $\omega \in \Omega^2(\mathbb{R}^3)$ given as the formal inner product between F and $\mathrm{d}S = (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$, i.e.

$$\omega = F \cdot \mathrm{d}S = F_x \,\mathrm{d}y \wedge \mathrm{d}z + F_y \,\mathrm{d}z \wedge \mathrm{d}x + F_z \,\mathrm{d}x \wedge \mathrm{d}y.$$

A direct computation then shows that

$$\mathrm{d}\omega = \Big(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\Big) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \mathrm{div}(F)\,\mathrm{d}V.$$

Hence by Stokes Theorem:

$$\int_A \mathrm{div}(F)\,\mathrm{d}V = \int_A \mathrm{d}\omega = \int_{\partial A} \omega = \int_{\partial A} F \cdot \mathrm{d}S$$

where of course all the forms in the integrals are understood to be restricted to A resp. ∂A. □