EPFL - Fall 2013	M. Troyanov
Introduction to smooth manifolds	January 13, 2014
Exam - English version	8:15–11:15

1. *(15 points)*

(a) Define the notion of a C^∞ cutoff function (fonction plateau) in a neighborhood of a point p in a smooth manifold M.

In the rest of this problem, we assume without proof that cutoff functions exist in the neighborhood of every point in M.

(b) Prove that $C^\infty(M)$ is a vector space and that $\dim(C^\infty(M)) = \infty$.

(c) Let $A, B \subset M$ be two nonempty compact subsets of M. Show that if $A \cap B = \emptyset$, then there exists a function $f \in C^\infty(M)$ such that $f \equiv 1$ on A and $f \equiv 0$ on B.

2. *(10 points)*

(a) Define the notion of a smooth map between two C^∞ manifolds.

(b) State the constant rank theorem.

(c) Explain what are immersions, embeddings and submersions.

(d) Prove that all submersions are open maps.

3. *(20 points)*

(a) Explain what is the tangent space at a point of a differentiable manifolds.

(b) What is its dimension? Justify your answer carefully!

(c) How can we associate a base of $T_p(M)$ to a local chart for M on a neighborhood of p ?

(d) Write down and prove the formula for the change of base on $T_p(M)$ when the underlying local chart on M changes.

4. *(15 points)*

(a) Give the definition of the exterior derivative d of a differential form.

(b) State four properties of d.

(c) Prove that for all forms α of degree 1, we have

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]).$$

5. *(10 points)*

Consider the smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = x^2 - y^2 .$$

For which values of $t \in \mathbb{R}$, the level set $f^{-1}(t)$ is a smooth submanifold of $\mathbb{R}^2$?

6. *(15 points)*
Consider the smooth manifold $\mathbb{R}^2 \setminus \{0\}$ with its standard structure, and define the vector fields

$$V = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}\,, \quad W = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\,.$$

(a) Compute the bracket $[V, W]$.

(b) What is the value of $[V, W]$ in polar coordinates ?

7. *(15 points)*
Let us denote by x, y, z the cartesian coordinates on $\mathbb{R}^3$ and by u, v the cartesian coordinates on $\mathbb{R}^2$, and consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\phi(u, v) = (u^2, uv, v^2)$. Compute the pullback of the following differential forms:

$$\phi^\star(dx)\,, \quad \phi^\star(dy)\,, \quad \phi^\star(dz)\,, \quad \phi^\star(dx \wedge dy \wedge dz)\,.$$

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8. Question facultative *(bonus max. 20 points)*

Let $U \subset \mathbb{R}^2$ be an open set on the Euclidean plane. We define an application $* : \Omega^1(U) \to \Omega^1(U)$ by

$$*\,(a(x,y)dx + b(x,y)dy) = b(x,y)dx - a(x,y)dy.$$

(a) What is $*(*\omega)$?

(b) Show that for all vectors $X \in T_pU$ and all forms $\omega \in \Omega^1(U)$ we have

$$(*\omega)(X) = \omega(\mathbf{J}X)$$

where $\mathbf{J}$ is the rotation operator of angle $+\pi/2$ in the positive sense, that is

$$\mathbf{J}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \qquad \mathbf{J}\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}.$$

(c) For a function $f \in C^\infty(U, \mathbb{R})$, compute $d{*}df$. How do we call a function such that $d{*}df = 0$?

(d) Let $f, g \in C^\infty(U, \mathbb{R})$, show that the function $h \in C^\infty(U, \mathbb{C})$ defined by

$$h = f + \sqrt{-1}g$$

is holomorphic if and only if $df = *dg$. Conclude that, in this case, $d * df = d * dg = 0$.

Remark: the map $*$ defined here is named the Hodge star operator *for differential forms in the plane.*