Homework 5: 29 March 2022
CS-526 Learning Theory

Exercise 1 (from S. Boyd & J. Duchi)

For the following convex functions, explain how to calculate a subgradient at a given $\mathbf{x}$.

1. $\forall \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = \max_{1 \le i \le m}(\mathbf{a}_i^T \mathbf{x} + b_i)$, where $\forall i \in \{1, \ldots, m\} : (\mathbf{a}_i, b_i) \in \mathbb{R}^n \times \mathbb{R}$.
2. $\forall \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = \max_{1 \le i \le m} |\mathbf{a}_i^T \mathbf{x} + b_i|$.
3. $\forall \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = \sup_{t \in [0,1]} p(t, \mathbf{x})$, where $p(t, \mathbf{x}) = x_1 + x_2 t + \cdots + x_n t^{n-1}$.

Exercise 2 (from S. Boyd & J. Duchi)

Convex functions that are not subdifferentiable. Verify that the following functions, defined on the interval $[0, +\infty)$, are convex, but not subdifferentiable at $x = 0$.

1. $f(0) = 1$, and $f(x) = 0$ for $x > 0$
2. $f(x) = -\sqrt{x}$

Exercise 3

We recall the definition of a strongly convex function:

Definition 1 *A function f is λ-strongly convex if for all $\mathbf{w}, \mathbf{u}$ and $\alpha \in (0, 1)$ we have:*

$$f(\alpha \mathbf{w} + (1 - \alpha)\mathbf{u}) \le \alpha f(\mathbf{w}) + (1 - \alpha) f(\mathbf{u}) - \frac{\lambda}{2}\alpha(1 - \alpha)\|\mathbf{w} - \mathbf{u}\|^2 .$$

Theorem 14.11 in the textbook is a refined bound for Stochastic Gradient Descent (SGD) when the function f is strongly convex. The proof of this theorem relies on the following claim (Claim 14.10 in *Understanding Machine Learning*):

Claim 1 *If f is λ-strongly convex then for every $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{v} \in \partial f(\mathbf{w})$ we have*

$$\langle \mathbf{w} - \mathbf{u}, \mathbf{v} \rangle \ge f(\mathbf{w}) - f(\mathbf{u}) + \frac{\lambda}{2}\|\mathbf{w} - \mathbf{u}\|^2$$

Prove this claim.

Exercise 4

Let $\pi_C(\mathbf{x}) = \arg\min_{\mathbf{y}\in C} \|\mathbf{x}-\mathbf{y}\|$ denote the Euclidean projection of x onto a closed convex set C of a Hilbert space H. Show that the projection is a 1-Lipschitz mapping, that is,

$$\|\pi_{\mathcal{C}}(\mathbf{x}_0) - \pi_C(\mathbf{x}_1)\| \leq \|\mathbf{x}_0 - \mathbf{x}_1\|\,,$$

for all vectors $\mathbf{x}_0, \mathbf{x}_1 \in H$. Show that the Lipschitz constant cannot be improved.
Hint: First prove the following important property of the projection onto a closed convex.

Lemma 1 *If C is a non-empty closed convex subset of a Hilbert space H then*

$$\forall(\mathbf{x},\mathbf{y}) \in H \times C : \langle \mathbf{x} - \pi_C(\mathbf{x}), \mathbf{y} - \pi_C(\mathbf{x})\rangle \leq 0\,.$$