Problem 1: Multilinear Rank, Tensor Rank

Recall the formulas for the matricizatons:
$T_{(1)} = A(C \otimes_{KhR} B)^T$, where A and $C \otimes_{KhR} B$ are of dimensions $I_1 \times R$ and $I_2 I_3 \times R$ respectively. Moreover for any matrices X, Y we have that:

$$\text{rank}(XY) \leq \min\{\text{rank}(X), \text{rank}(Y)\}$$

Thus:

$$R_1 = \text{rank}(T_{(1)}) \leq \text{rank}(A) \leq \min\{I_1, R\} \leq R$$

By repeating the same argument for matrizications $T_{(2)}$, $T_{(3)}$ we conclude the proof.

Problem 2: Non-unicity of Tucker decomposition

Let $X = [\vec{x_1}, \ldots, \vec{x_{R_1}}]$, $Y = [\vec{y_1}, \ldots, \vec{y_{R_2}}]$, and $Z = [\vec{z_1}, \ldots, \vec{z_{R_3}}]$. Then, from the definitions of vectors $\vec{x_{p'}}$, $\vec{y_{q'}}$, $\vec{z_{r'}}$, and from the orthogonality of the matrices $M^{(u)}, M^{(v)}, M^{(w)}$ it is easy to see that:

1. $U \cdot (M^{(u)})^T = X \Rightarrow U = X \cdot M^{(u)} \Rightarrow \vec{u}_p = X \cdot M^{(u)}_{:p} = \sum_{p'} M^{(u)}_{p'p} \vec{x}_{p'}$
2. $V \cdot (M^{(v)})^T = Y \Rightarrow V = Y \cdot M^{(v)} \Rightarrow \vec{v}_q = Y \cdot M^{(v)}_{:q} = \sum_{q'} M^{(v)}_{q'q} \vec{y}_{q'}$
3. $W \cdot (M^{(w)})^T = Z \Rightarrow W = Z \cdot M^{(w)} \Rightarrow \vec{w}_r = Z \cdot M^{(w)}_{:r} = \sum_{r'} M^{(w)}_{r'r} \vec{z}_{r'}$

Substituting $\vec{u}_p, \vec{v}_q$ and $\vec{w}_r$ in the Tucker decomposition expression we get:

$$T = \sum_{p,q,r=1}^{R_1,R_2,R_3} G^{pqr}\, \vec{u}_p \otimes \vec{v}_q \otimes \vec{w}_r = \sum_{p,q,r=1}^{R_1,R_2,R_3} G^{pqr}\, (\sum_{p'} M^{(u)}_{p'p} \vec{x}_{p'}) \otimes (\sum_{q'} M^{(v)}_{q'q} \vec{y}_{q'}) \otimes (\sum_{r'} M^{(w)}_{r'r} \vec{z}_{r'}) =$$

$$\sum_{p',q',r'=1}^{R_1,R_2,R_3} \sum_{p,q,r=1}^{R_1,R_2,R_3} G^{pqr} M^{(u)}_{p'p} M^{(v)}_{q'q} M^{(w)}_{r'r}\, \vec{x}_{p'} \otimes \vec{y}_{q'} \otimes \vec{z}_{r'} = \sum_{p',q',r'=1}^{R_1,R_2,R_3} H^{p'q'r'}\, \vec{x}_{p'} \otimes \vec{y}_{q'} \otimes \vec{z}_{r'}$$

where $H^{p'q'r'} = \sum_{p,q,r=1}^{R_1,R_2,R_3} G^{pqr} M^{(u)}_{p'p} M^{(v)}_{q'q} M^{(w)}_{r'r}$, which concludes the proof.

Problem 3: Tensor decomposition & estimation of a sensing matrix

1. The empirical estimate $\widehat{S} = \frac{1}{L} \sum_{\ell=1}^{L} \vec{y}^{(\ell)} \otimes \vec{y}^{(\ell)}$ of S has components

$$\widehat{S}_{\alpha\beta} = \frac{1}{L} \sum_{\ell=1}^{L} y^\ell_\alpha y^\ell_\beta \, .$$

The empirical estimate $\widehat{F} = \frac{1}{L} \sum_{\ell=1}^{L} \vec{y}^{(\ell)} \otimes \vec{y}^{(\ell)} \otimes \vec{y}^{(\ell)} \otimes \vec{y}^{(\ell)}$ of F has components

$$\widehat{F}_{\alpha\beta\gamma\delta} = \frac{1}{L} \sum_{\ell=1}^{L} y^\ell_\alpha y^\ell_\beta y^\ell_\gamma y^\ell_\delta \, .$$

2. Remember that $\vec{y} = \sum_{k=1}^{k} h_k \vec{\mu}^{(k)}$. By expanding the tensor products we get:

$$S = \mathbb{E}[\vec{y} \otimes \vec{y}] = \sum_{j=1}^{K} \sum_{k=1}^{K} \mathbb{E}[h_j h_k] \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \; ; \tag{1}$$

$$F = \mathbb{E}[\vec{y} \otimes \vec{y} \otimes \vec{y} \otimes \vec{y}] = \sum_{i=1}^{K} \sum_{j=1}^{K} \sum_{k=1}^{K} \sum_{\ell=1}^{K} \mathbb{E}[h_i h_j h_k h_\ell] \vec{\mu}^{(i)} \otimes \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(\ell)} \; . \tag{2}$$

The components of h are independent and centered so $\mathbb{E}[h_j h_k] = \mathbb{E}[h_j]\mathbb{E}[h_k] = 0$ if $j \neq k$. Hence (1) simplifies:

$$S = \mathbb{E}[\vec{y} \otimes \vec{y}] = \sum_{k=1}^{K} \mathbb{E}[h_k^2] \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \; .$$

Similarly, if one of the indices i, j, k, ℓ is distinct of all the others then $\mathbb{E}[h_i h_j h_k h_\ell] = 0$. Therefore $\mathbb{E}[h_i h_j h_k h_\ell]$ is nonzero if, and only if, $i = j = k = \ell$ or $i = j \neq k = \ell$, $i = k \neq j = \ell$, $i = \ell \neq j = k$. Hence (2) reads:

$$\begin{aligned} F = \sum_{k=1}^{K} \mathbb{E}[h_k^4] \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} &+ \sum_{1 \le i \neq k \le K} \mathbb{E}[h_i^2]\mathbb{E}[h_k^2] \vec{\mu}^{(i)} \otimes \vec{\mu}^{(i)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \\ &+ \sum_{1 \le k \neq j \le K} \mathbb{E}[h_k^2]\mathbb{E}[h_j^2] \vec{\mu}^{(k)} \otimes \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(j)} \\ &+ \sum_{1 \le i \neq k \le K} \mathbb{E}[h_i^2]\mathbb{E}[h_k^2] \vec{\mu}^{(i)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(i)} \\ = \sum_{k=1}^{K} \mathbb{E}[h_k^4] \, \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} &+ \sum_{1 \le j \neq k \le K} \mathbb{E}[h_j^2]\mathbb{E}[h_k^2] \Big(\vec{\mu}^{(j)} \otimes \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \\ &\qquad + \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \\ &\qquad + \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(j)} \Big) . \end{aligned}$$

3. We have seen that $F = \sum_{k=1}^{K} \mathbb{E}[h_k^4] \, \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} + E$ where

$$\begin{aligned} E := \sum_{1 \le j \neq k \le K} \mathbb{E}[h_j^2]\mathbb{E}[h_k^2] \Big(& \vec{\mu}^{(j)} \otimes \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} + \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \\ & + \vec{\mu}^{(j)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(j)} \Big) \; . \end{aligned}$$

The components of E satisfy:

$$\begin{aligned} E_{\alpha\beta\gamma\delta} &= \sum_{1 \le j \neq k \le K} \mathbb{E}[h_j^2]\mathbb{E}[h_k^2] (\mu_\alpha^j \mu_\beta^j \mu_\gamma^k \mu_\delta^k + \mu_\alpha^j \mu_\beta^k \mu_\gamma^j \mu_\delta^k + \mu_\alpha^j \mu_\beta^k \mu_\gamma^k \mu_\delta^j) \\ &= \left(\sum_{j=1}^{K} \mathbb{E}[h_j^2] \mu_\alpha^j \mu_\beta^j \right) \left(\sum_{k=1}^{K} \mathbb{E}[h_k^2] \mu_\gamma^k \mu_\delta^k \right) + \left(\sum_{j=1}^{K} \mathbb{E}[h_j^2] \mu_\alpha^j \mu_\gamma^j \right) \left(\sum_{k=1}^{K} \mathbb{E}[h_k^2] \mu_\beta^k \mu_\delta^k \right) \\ &\quad + \left(\sum_{j=1}^{K} \mathbb{E}[h_j^2] \mu_\alpha^j \mu_\delta^j \right) \left(\sum_{k=1}^{K} \mathbb{E}[h_k^2] \mu_\beta^k \mu_\gamma^k \right) - 3 \sum_{k=1}^{K} \mathbb{E}[h_k^2]^2 \mu_\alpha^k \mu_\beta^k \mu_\gamma^k \mu_\delta^k \\ &= S_{\alpha\beta} S_{\gamma\delta} + S_{\alpha\gamma} S_{\beta\delta} + S_{\alpha\delta} S_{\beta\gamma} - 3 \left(\sum_{k=1}^{K} \mathbb{E}[h_k^2]^2 \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \right)_{\alpha\beta\gamma\delta} . \end{aligned}$$

It directly follows that

$$T_{\alpha\beta\gamma\delta} = \left(\sum_{k=1}^{K} \mathbb{E}[h_k^4] \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \right)_{\alpha\beta\gamma\delta} - 3 \left(\sum_{k=1}^{K} \mathbb{E}[h_k^2]^2 \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \right)_{\alpha\beta\gamma\delta}$$

$$= \left(\sum_{k=1}^{K} \mathcal{K}_k \mathbb{E}[h_k^2]^2 \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \otimes \vec{\mu}^{(k)} \right)_{\alpha\beta\gamma\delta} .$$

4. The matrix S is symmetric positive semidefinite so it can be diagonalised in an orthonormal basis: $S = \sum_{k=1}^{N} d_k \vec{u}^{(k)} \otimes \vec{u}^{(k)}$ with $\begin{bmatrix} \vec{u}^{(1)} & \vec{u}^{(2)} & \cdots & \vec{u}^{(N)} \end{bmatrix} \in \mathbb{R}^{N \times N}$ an orthonormal matrix and $d_1 \geq d_2 \geq \cdots \geq d_N \geq 0$. Besises S has rank K so exactly K of its eigenvalues are nonzero: $S = \sum_{k=1}^{K} d_k \vec{u}^{(k)} \otimes \vec{u}^{(k)} = UDU^T$ with $U = \begin{bmatrix} \vec{u}^{(1)} & \vec{u}^{(2)} & \cdots & \vec{u}^{(K)} \end{bmatrix} \in \mathbb{R}^{N \times K}$ and $D = \mathrm{Diag}(d_1, d_2, \ldots, d_K)$.

5. By definition of the vectors $\vec{v}^{(k)}$ we have:

$$\widetilde{T} = \sum_{k=1}^{K} \mathcal{K}_k \mathbb{E}[h_k^2]^2 (W^T \vec{\mu}^{(k)}) \otimes (W^T \vec{\mu}^{(k)}) \otimes (W^T \vec{\mu}^{(k)}) \otimes (W^T \vec{\mu}^{(k)}) .$$

   So the components of $\widetilde{T}$ are given by the formula

$$\widetilde{T}_{\alpha\beta\gamma\delta} = \sum_{\alpha',\beta',\gamma',\delta'} W_{\alpha'\alpha} W_{\beta'\beta} W_{\gamma'\gamma} W_{\delta'\delta} \, T_{\alpha'\beta'\gamma'\delta'} .$$

   This transformation of T into $\widetilde{T}$ is called a whitening process.

6. If $\mathcal{K}_k$ is zero then it is impossible to recover $\vec{v}^{(k)}$ with the tensor power method and, as a consequence, to recover $\vec{\mu}^{(k)}$ the k^{th} column of A.

7. The pseudocode for the tensor power method is given in Algorithm 1.

8. By definition $\vec{v}^{\,(k)} = \sqrt{\mathbb{E}[h_k^2]} W^T \vec{\mu}^{(k)} = \sqrt{\mathbb{E}[h_k^2]} D^{-\frac{1}{2}} U^T \vec{\mu}^{(k)}$ so

$$UU^T \vec{\mu}^{(k)} = UD^{\frac{1}{2}} \frac{\vec{v}^{\,(k)}}{\sqrt{\mathbb{E}[h_k^2]}} .$$

   Finally, $\vec{\mu}^{(k)}$ belongs to the subspace spanned by the columns of U so $UU^T \vec{\mu}^{(k)} = \vec{\mu}^{(k)}$. We conclude that $\vec{\mu}^{(k)} = UD^{\frac{1}{2}} \vec{v}^{\,(k)} / \sqrt{\mathbb{E}[h_k^2]}$. Of course, we only know $\vec{v}^{\,(k)}$ up to a plus or minus sign and we will recover $\vec{\mu}^{(k)}$ up to a plus or minus sign too.

Algorithm 1 Tensor power method

procedure POWERMETHOD($\widetilde{T}, K, T_{\max}$)
    vectors $\leftarrow$ [ ]
    kurtoses $\leftarrow$ [ ]
    for $i = 1$ **to** K **do**
        $v \leftarrow$ normallyDistributedVector(size $= K$)
        for $j = 1$ **to** $T_{\max}$ **do** ▷ $T_{\max}$ iterations of the power method.
            for $\alpha = 1$ **to** K **do**
                $v[\alpha] \leftarrow \sum_{\beta,\gamma,\delta=1}^{K} \widetilde{T}_{\alpha,\beta,\gamma,\delta} v[\beta] v[\gamma] v[\delta]$
            end for
            $v \leftarrow v/\|v\|$ ▷ Renormalizing v
        end for
        $\mathcal{K} \leftarrow \sum_{,\alpha,\beta,\gamma,\delta=1}^{K} \widetilde{T}_{\alpha,\beta,\gamma,\delta} v[\alpha] v[\beta] v[\gamma] v[\delta]$
        kurtoses.*append*($\mathcal{K}$)
        vectors.*append*(v)
        $\widetilde{T} \leftarrow \widetilde{T} - \mathcal{K}\, v^{\otimes 4}$ ▷ Subtracting the recovered rank-one tensor from $\widetilde{T}$
    end for
    return vectors, kurtoses
end procedure