EPFL - Printemps 2022	Prof. Z. Patakfalvi
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Solution de l'Exercice bonus 1	

1 Exercice Bonus 1

Exercice 7. 1. $\nu(1) = \nu(1 \cdot 1) = \nu(1) + \nu(1)$. Therefore $\nu(1) = 0$. The case of -1 follows from this by $\nu(1) = \nu(-1 \cdot -1) = \nu(-1) + \nu(-1)$.

2. We need to prove that R_ν is table under addition and multiplication and that the neutral element 1 and the zero element 0 belong to R_ν.

   a) From point 1. we have that $1 \in R_\nu$.

   b) Note that $0 \in R_\nu$ by definition of R_ν.

   c) Let $x, y \in R_\nu$, then $\nu(x \cdot y) = \nu(x) + \nu(y) \geq 0$. Therefore $x \cdot y \in R_\nu$.

   d) Let $x, y \in R_\nu$, then $\nu(x + y) \geq \min(\nu(x), \nu(y)) \geq 0$. Therefore $x + y \in R_\nu$.

3. We know from the previous point that R_ν is a subring of K. We will prove that for all $x \in K$ either $x \in K$ or $x^{-1} \in K$. This will prove that K is the fraction field of R_ν. From point 1. we have that
$$0 = \nu(1) = \nu(x \cdot x^{-1}) = \nu(x) + \nu(x^{-1}).$$
Thus $\nu(x) = -\nu(x^{-1})$. It follows that either $\nu(x) \in R_\nu$ or $\nu(x^{-1}) \in R_\nu$.

4. From now on we will consider $K = \mathbb{Q}$. We want to prove that if $x \in \mathbb{Z}$ then $\nu(x) \geq 0$. Using point 1. we have that $\nu(1 + ... + 1) \geq \min(\nu(1), ..., \nu(1)) = 0$. The case of negative numbers is similar.

5. Suppose that $\nu(p) = 0$ for all primes p, then ν is trivial. Then for every $x \in \mathbb{Z}$ we consider it prime factorization $z = p_1^{j_1} p_2^{j_2} \cdot p_n^{j_n}$ where p_i are prime and $j_i \in \mathbb{N}$. Then using iteratevly condition a. in the definition of valuation function we first note that $\nu(p_i^{n_i}) = 0$ for every $i \in 1 \ldots n$. Again using condition a. we can conclude that $\nu(z) = 0$. Therefore ν is trivial.

6. Now we want to prove that if ν is non-trivial, then $\nu(p) \neq 0$ can happen for at most one (positive) prime p. Let p and q are two distinct such primes, then by the Euclidean algorithm we can write $1 = ap + bq$ for a and b integers. Suppose by contradiction that both $\nu(p) \geq 0$ and $\nu(q) \geq)$. Then by condition a. and b. above we have
$$0 = \nu(1) = \nu(ap + bq) \geq \min(\nu(ap), \nu(bq)).$$
Suppose without loss of generality that $0 \geq \nu(ap) = \nu(a) + \nu(p)$. We know by point 4. that $\nu(a) \geq 0$ and $\nu(p) \geq 0$. Then we have necessarily that $\nu(p) = 0$ and $\nu(a) = 0$. Therefore we proved our statement by contradiction.

7. Let ν a non-trivial valuation function such that $\nu(p) \neq 0$ for a prime p. We know that $\nu(p^i) \geq 0$ by condition a.. The for every $x \in \mathbb{Q}$ we can write $x = p^i \frac{a}{b}$, where a and b coprime with p. Then we have that
$$\nu(p^i \frac{a}{b}) = \nu(p^i) + \nu(\frac{a}{b}) = \nu(p^i) = \nu(p) + \nu(p) + \cdots + \nu(p) = i \cdot c,$$
where the last equality is given by the fact that $\nu(p)$ is summed i-times and we defined $c := \nu(p)$.
It is not difficult to prove that
$$\nu(p^i \frac{a}{b}) = i \cdot c,$$

is a valuation function. Where $\nu(p) \geq 0$, a, b prime with p and c is an integer. In fact we have that for every $x, y \in \mathbb{Z}$ such that $x = p^i \frac{a}{b}$ and $y = p^j \frac{c}{d}$, with a, b, c, d prime with p:

a. $\nu(xy) = \nu(p^{i+j} \frac{a}{b} \frac{c}{d}) = (i+j)c = \nu(x) + \nu(y)$.

b. One can show that $\nu(x+y) \geq k \cdot c$ where $k =\geq \min(i, j)$ by taking a prime decomposition of the sum $x + y$ and noticing that at least the the minimum between p^i or p^j has to appear in this factorization.

8. The above valuation function is called the p-adic valuation for $c = 1$, which we denote by ν_p. We want to show that the valuation ring on R_{ν_p} is not equal to $\mathbb{Z} \subset \mathbb{Q}$. One can see that $\nu_p(p^i/q) \geq 0$ for every prime q other than p, thereforeR_{ν_p} is not equal to $\mathbb{Z} \subset \mathbb{Q}$.

Note that with proving all the points above we exhibited $\mathbb{Q}$ as the fraction field of many subrings other than $\mathbb{Z}$ built as the evaluation R_{ν_p} for some prime p.