Low-power radio design for the IoT
Exercise 2 (03.03.2022)

Christian Enz

Swiss Federal Institute of Technology (EPFL), Lausanne, Switzerland

Problem 1 Phase Mismatch in 16QAM Generator

Due to imperfections, a 16QAM generator produces:

$$x(t) = \alpha_1 A_c \cos(\omega_c t + \Delta\theta) - \alpha_2 A_c (1 + \epsilon) \sin(\omega_c t). \tag{1}$$

where $\alpha_1 = \pm 1, \pm 2$ and $\alpha_2 = \pm 1, \pm 2$.

- Construct the signal constellation for $\Delta\theta \neq 0$ but $\epsilon = 0$.
- Construct the signal constellation for $\Delta\theta = 0$ but $\epsilon \neq 0$.

Problem 2 Spectral Regrowth and Transmission Mask Requirements

A two-tone signal $x(t) = A_1 \cos\omega_1 t + A_2 \cos\omega_2 t$ is the input of a nonlinear power amplifier (PA) with compressing characteristic $y(t)$:

$$y(t) = \alpha_1 x(t) - |\alpha_3| x^3(t) \tag{2}$$

The spectrum of the signal at the output of the PA must respect the transmission mask shown in Fig. 1, where $\Delta f = 0$ corresponds to the center of the band of the input signal.

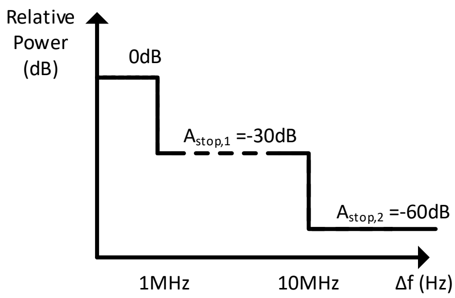

Figure 1: Transmission Mask

Find the value for $|\alpha_3|$ that allows to respect such mask for $\omega_1 = 2\pi * 2.409\,\text{GHz}$, $\omega_2 = 2\pi * 2.41\,\text{GHz}$ and $A_1 = A_2 = 0\,\text{dBm}$.

Problem 3 BER for M-ASK signaling

In the lecture it was shown that the probability of error for the BPSK signal is given by:

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right),\ E_b = \frac{A^2 T_b}{2} \tag{3}$$

for a constellation given in Fig. 2(a). Figure 2(b) shows the constellation points of the 4-ASK modulation.

- Determine the average energy per symbol for the 4-ASK modulation.
- Derive the probability of error for the 4-ASK modulation starting from the expression for the BPSK.
- Derive the probability of error for the general case of M-ASK modulation. Assume the constellation points are given by

$$x_{m1,2} = \pm\frac{2m-1}{2}A,\ m \in [1, M/2]\,. \tag{4}$$

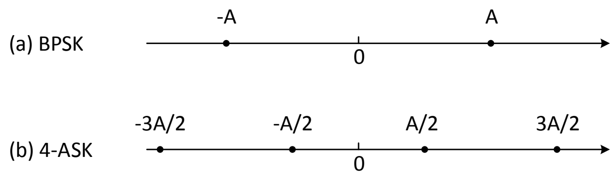

Figure 2: Constellation points of (a) BPSK modulation and (b) 4-ASK modulation.

Solutions to Exercise 2 (03.03.2022)

Problem 1 Phase Mismatch in 16QAM Generator

- Construct the signal constellation for $\Delta\theta \neq 0$ but $\epsilon = 0$.

Using the trigonometric identity $\cos \alpha \pm \beta = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$, $x(t)$ can be reduced in the form $x(t) = \beta_1 A_c \cos(\omega_c t) - \beta_2 A_c \sin(\omega_c t)$ as:

$$\begin{aligned} x(t) &= \alpha_1 A_c \cos(\omega_c t + \Delta\theta) - \alpha_2 A_c \sin(\omega_c t) \\ &= \alpha_1 A_c (\cos(\omega_c t) \cos\Delta\theta - \sin(\omega_c t)\sin\Delta\theta) - \alpha_2 A_c \sin(\omega_c t) \\ &= \alpha_1 A_c \cos\Delta\theta \cos(\omega_c t) - (\alpha_1 A_c \sin\Delta\theta + \alpha_2 A_c)\sin(\omega_c t). \end{aligned}$$

Depending on the value of the couple $[\alpha_1\ \alpha_2]$, there are 16 combinations for $[\beta_1\ \beta_2]$:

$$\begin{aligned}
[1\ 1] &\Longrightarrow [\cos\Delta\theta \quad -(1+\sin\Delta\theta)] \\
[1\ 2] &\Longrightarrow [\cos\Delta\theta \quad -2(2+\sin\Delta\theta)] \\
[2\ 1] &\Longrightarrow [2\cos\Delta\theta \quad -(1+2\sin\Delta\theta)] \\
[2\ 2] &\Longrightarrow [2\cos\Delta\theta \quad -2(1+\sin\Delta\theta)] \\
[-1\ 1] &\Longrightarrow [-\cos\Delta\theta \quad -(1+\sin\Delta\theta)] \\
[-1\ 2] &\Longrightarrow [-\cos\Delta\theta \quad -(2+\sin\Delta\theta)] \\
[-2\ 1] &\Longrightarrow [-2\cos\Delta\theta \quad -(1+2\sin\Delta\theta)] \\
[-2\ 2] &\Longrightarrow [-2\cos\Delta\theta \quad -2(1+\sin\Delta\theta)] \\
[1\ \ -1] &\Longrightarrow [\cos\Delta\theta \quad 1-\sin\Delta\theta] \\
[1\ \ -2] &\Longrightarrow [\cos\Delta\theta \quad 2-\sin\Delta\theta] \\
[2\ \ -1] &\Longrightarrow [2\cos\Delta\theta \quad 1-2\sin\Delta\theta] \\
[2\ \ -2] &\Longrightarrow [2\cos\Delta\theta \quad 2(1-\sin\Delta\theta)] \\
[-1\ \ -1] &\Longrightarrow [-\cos\Delta\theta \quad 1-\sin\Delta\theta] \\
[-1\ \ -2] &\Longrightarrow [-\cos\Delta\theta \quad 2-\sin\Delta\theta] \\
[-2\ \ -1] &\Longrightarrow [-2\cos\Delta\theta \quad 1-2\sin\Delta\theta] \\
[-2\ \ -2] &\Longrightarrow [-2\cos\Delta\theta \quad 2(1-\sin\Delta\theta)]
\end{aligned}$$

Fig. 1a) shows how the constellation is deformed due to $\Delta\theta$.

- Construct the signal constellation for $\Delta\theta = 0$ but $\epsilon \neq 0$.

$x(t)$ is already in the form $x(t) = \beta_1 A_c \cos(\omega_c t) - \beta_2 A_c \sin(\omega_c t)$. Again, depending on the value of the

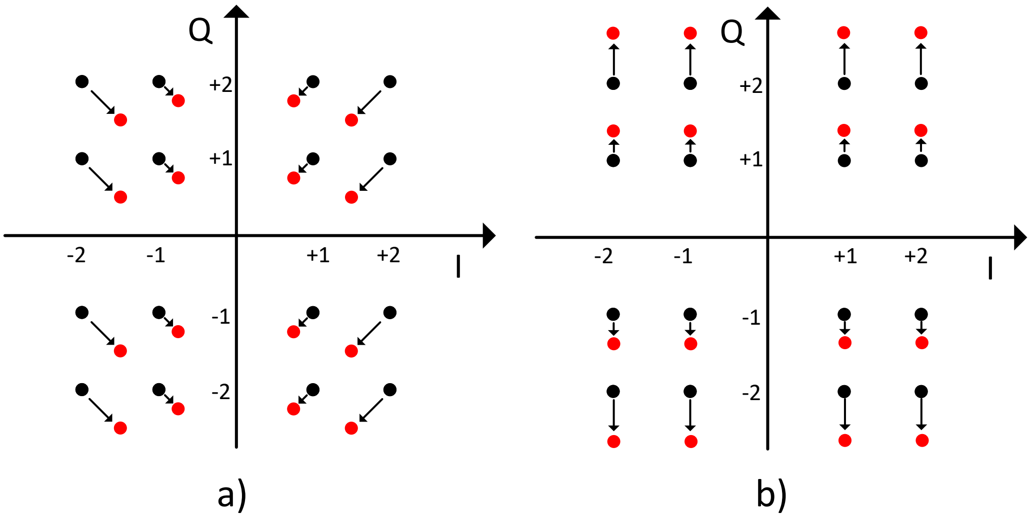

Figure 1: 16QAM constellation deformed by $\Delta\theta$

couple $[\alpha_1\ \alpha_2]$, there are 16 combinations for $[\beta_1\ \beta_2]$:

$$
\begin{aligned}
[1\ 1] &\Longrightarrow [1 \quad -(1+\epsilon)]\\
[1\ 2] &\Longrightarrow [1 \quad -2(1+\epsilon)]\\
[2\ 1] &\Longrightarrow [2 \quad -(1+\epsilon)]\\
[2\ 2] &\Longrightarrow [2 \quad -2(1+\epsilon)]\\
[-1\ 1] &\Longrightarrow [-1 \quad -(1+\epsilon)]\\
[-1\ 2] &\Longrightarrow [-1 \quad -2(1+\epsilon)]\\
[-2\ 1] &\Longrightarrow [-2 \quad -(1+\epsilon)]\\
[-2\ 2] &\Longrightarrow [-2 \quad -2(1+\epsilon)]\\
[1\ -1] &\Longrightarrow [1 \quad (1+\epsilon)]\\
[1\ -2] &\Longrightarrow [1 \quad 2(1+\epsilon)]\\
[2\ -1] &\Longrightarrow [2 \quad (1+\epsilon)]\\
[2\ -2] &\Longrightarrow [2 \quad 2(1+\epsilon)]\\
[-1\ -1] &\Longrightarrow [-1 \quad (1+\epsilon)]\\
[-1\ -2] &\Longrightarrow [-1 \quad 2(1+\epsilon)]\\
[-2\ -1] &\Longrightarrow [-2 \quad (1+\epsilon)]\\
[-2\ -2] &\Longrightarrow [-2 \quad 2(1+\epsilon)]
\end{aligned}
$$

Fig. 1b) shows how the constellation is deformed due to ϵ.

Problem 2 Spectral Regrowth and Transmission Mask Requirements

- Find the value for $|\alpha_3|$ that allows to respect such mask for $\omega_1 = 2\pi * 2.409\,\text{GHz}$, $\omega_2 = 2\pi * 2.41\,\text{GHz}$ and $A_1 = A_2 = 0\,\text{dBm}$.

  First, the expression of the output $y(t)$ in terms of its spectral components has to be derived by replacing

$x(t)$ with the given expression:

$$\begin{aligned}y(t) =&\alpha_1 x(t) - |\alpha_3| x^3(t) = \alpha_1 (A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)) - |\alpha_3|(A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t))^3 \\ =&\alpha_1 A_1 \cos(\omega_1 t) + \alpha_1 A_2 \cos(\omega_2 t) - |\alpha_3|(A_1^3 \cos^3(\omega_1 t) + 3A_1^2 A_2 \cos^2(\omega_1 t)\cos(\omega_2 t) \\ &+ 3A_1 A_2^2 \cos(\omega_1 t)\cos^2(\omega_2 t) + A_2^3 \cos^3(\omega_2 t))\end{aligned}$$

Using the trigonometrical identities $\cos^3\alpha = (\cos(3\alpha) + 3\cos\alpha)/4$, $\cos^2\alpha = (\cos(2\alpha)+1)/2$ and $\cos\alpha\cos\beta = (\cos(\alpha-\beta) + \cos(\alpha+\beta))/2$, $y(t)$ can be simplified as follows:

$$\begin{aligned}y(t) =&\alpha_1 A_1 \cos(\omega_1 t) + \alpha_1 A_2 \cos(\omega_2 t) - |\alpha_3|(A_1^3 \frac{\cos(3\omega_1 t) + 3\cos(\omega_1 t)}{4} + 3A_1^2 A_2 \frac{\cos(2\omega_1 t)+1}{2}\cos(\omega_2 t) \\ &+ 3A_1 A_2^2 \cos(\omega_1 t)\frac{\cos(2\omega_2 t)+1}{2} + A_2^3 \frac{\cos(3\omega_2 t) + 3\cos(\omega_2 t)}{4}) \\ =&(\alpha_1 A_1 - |\alpha_3|(\frac{3}{4}A_1^3 + \frac{3}{2}A_1 A_2^2))\cos(\omega_1 t) + (\alpha_1 A_2 - |\alpha_3|(\frac{3}{4}A_2^3 + \frac{3}{2}A_1^2 A_2))\cos(\omega_2 t) \\ &- \frac{1}{4}|\alpha_3| A_1^3 \cos(3\omega_1 t) - \frac{1}{4}|\alpha_3| A_2^3 \cos(3\omega_2 t) \\ &- \frac{3}{4}|\alpha_3| A_1^2 A_2 \cos((2\omega_1 - \omega_2)t) - \frac{3}{4}|\alpha_3| A_1 A_2^2 \cos((2\omega_2 - \omega_1)t) \\ &- \frac{3}{4}|\alpha_3| A_1^2 A_2 \cos((2\omega_1 + \omega_2)t) - \frac{3}{4}|\alpha_3| A_1 A_2^2 \cos((2\omega_2 + \omega_1)t).\end{aligned}$$

Because of the nonlinear characteristic of the PA, the output contains additional terms at the fundamental frequency which compress the gain and additionally there are components at the third harmonics and at the third-order intermodulation products.

In order to estimate the value of $|\alpha_3|$ that allows to respect the given mask for all the components in the output, we first need to convert A_1, A_2 and the maximum tolerated amplitude in the stop bands ($A_{\text{stop},1}$ and $A_{\text{stop},2}$) from dBm to volt using the following formulas:

$$P = \frac{V_\text{p}^2}{2 * 50\,\Omega} \quad P_\text{dBm} = 10\log_{10}\frac{P}{1\,\text{mW}}$$

$$\begin{aligned}A_{1,2} =& 0\,\text{dBm} \Longrightarrow 316.2\,\text{mV} \\ A_{\text{stop},1} =& -30\,\text{dBm} \Longrightarrow 10\,\text{mV} \\ A_{\text{stop},2} =& -60\,\text{dBm} \Longrightarrow 316.2\,\mu\text{V}\end{aligned}$$

The IM products at $2\omega_1 - \omega_2$ and $2\omega_2 - \omega_1$ fall at 2.408 GHz and 2.411 GHz respectively and they have to be smaller than $A_{\text{stop},1}$ since they are more than 1 MHz far from the center of the band of the input signal, namely 2.4095 GHz. On the other hand, the IM products at $2\omega_1 + \omega_2$ and $2\omega_2 + \omega_1$ fall at 7.228 GHz and 7.229 GHz and the third harmonics at 7.227 GHz and 7.23 GHz: all of them have to be smaller than $A_{\text{stop},2}$.

Given that $A_1 = A_2 = A$, 3 different values for $|\alpha_3|$ can be extracted:

$$\begin{aligned}2\omega_1 - \omega_2, 2\omega_2 - \omega_1 \Longrightarrow& |\alpha_3| = 0.42\,\text{V}^{-2} \\ 2\omega_1 + \omega_2, 2\omega_2 + \omega_1 \Longrightarrow& |\alpha_3| = 0.013\,\text{V}^{-2} \\ 3\omega_1, 3\omega_2 \Longrightarrow& |\alpha_3| = 0.04\,\text{V}^{-2}\end{aligned}$$

The condition on the third-order IM products $2\omega_1 + \omega_2$ and $2\omega_2 + \omega_1$ is the most stringent and the corresponding value for $|\alpha_3|$ has to be selected.

Problem 3 BER for M-ASK signaling

- Determine the average energy per symbol for the 4-ASK modulation. The average energy per symbol can be calculated as

$$\overline{E_s} = \frac{1}{M}\sum_{m=1}^{M} E_m. \tag{1}$$

For the case of 4-ASK modulation the average energy per symbol is then given by

$$\overline{E_s} = \frac{1}{4}\left(2\frac{A^2 T_s}{8} + 2\frac{9A^2 E_s}{8}\right) = \frac{5}{8}A^2 T_s. \tag{2}$$

- Derive the probability of error for the 4-ASK modulation starting from the expression for the BPSK. For the two outer constellation points the probability of error is given by

$$P_{e,out} = Q\left(\sqrt{\frac{A^2 T_s}{4N_0}}\right). \tag{3}$$

For the two inner constellation points the probability of error is doubled since there is another point on each side:

$$P_{e,in} = 2Q\left(\sqrt{\frac{A^2 T_s}{4N_0}}\right). \tag{4}$$

The average probability of error is then given by

$$P_e = 2\left(\frac{1}{4}Q\left(\sqrt{\frac{A^2 T_s}{4N_0}}\right) + \frac{2}{4}Q\left(\sqrt{\frac{A^2 T_s}{4N_0}}\right)\right), \tag{5}$$

$$P_e = \frac{3}{2}Q\left(\sqrt{\frac{A^2 T_s}{4N_0}}\right). \tag{6}$$

- Derive the probability of error for the general case of M-ASK modulation. Assume the constellation points are given by

$$x_{m1,2} = \pm\frac{2m-1}{2}A,\ m \in [1, M/2]\,. \tag{7}$$

For the general case of M-ASK there will be two outer points and M-2 inner points (points with two neighboring points) in the constellation. The expression for the probability of error is then given by

$$P_e = \frac{2}{M}Q\left(\sqrt{\frac{A^2 T_s}{4N_0}}\right) + \frac{2(M-2)}{M}Q\left(\sqrt{\frac{A^2 T_s}{4N_0}}\right), \tag{8}$$

$$P_e = \frac{2(M-1)}{M}Q\left(\sqrt{\frac{A^2 T_s}{4N_0}}\right). \tag{9}$$