# 1 Exercice Bonus 2

**Exercice 8.** 1. By the definition of a valuation we have that $\nu_p(q^{-1}) = 0$ too, because $\nu_p(q) + \nu_p(q^{-1}) = \nu_p(1) = 0$. Therefore $q^{-1} \in R$ and $q$ is invertible.

2. The zero ideal is trivially an ideal of $R$. Now, take a non-zero ideal $I$, and let $n$ be the smallest valuation that appears among the elements of $I$. Then there is an element of the form $y = p^n q$, where $q$ is a unit. Take now any $x \in I$ non-zero. Then $\nu_p(x/y) \geq 0$, again by the properties of valuations and by the minimality of $n$, hence $x/y \in R$, and hence $I = (y)$

3. Consider the composition $\phi : \mathbb{Z} \to R \to R/(p^n)$. Where the first map is the inclusion and the second one is the quotient map. Then we can apply the isomorphism theorem to $\mathbb{Z}$, because

   a) $\phi$ is surjective because let $a/b \in R$ (with $a, b \in \mathbb{Z}$ and $p \nmid b$). Then we can write $cp^n + db = 1$ for some $c, d \in \mathbb{Z}$. Hence $[d][b] = [1] \in R/(p^n)$. Hence, for every $[ab^{-1}]$ in $R/(p^n)$ we have $[a][b^{-1}] = [a][b^{-1}][b][d] = [a][d] = [ad] = \phi(ad)$.

   b) The kernel of $\phi$ is generated by $p^n$ as an ideal of $\mathbb{Z}$, because if $x$ is in the kernel, that means that $x = (a/b)p^n \in R$, where $a$ and $b$ are as in the previous point. That is, $bx = ap^n$. Now, using that $p \nmid b$ we obtain that $p^n$ divides $x \in \mathbb{Z}$.

4. From the previous points we know all the non-trivial ideals of $R_p$ are of the form $(p^n)$ for some $n \in \mathbb{N}$, and we know that their quotient is isomorphic to $\mathbb{Z}/(p^n)$. If $R_p$ and $R_q$ were isomorphic for two different prime numbers $p$ and $q$, there would be isomorphism between their quotients. This is impossible because $\mathbb{Z}/p^m Z$ and $\mathbb{Z}/q^n Z$ are not isomorphic since their size is different ($p^m$ and $q^n$ respectively). Moreover, $R_p$ is not isomorphic to $\mathbb{Z}$. To show this note that for every $q$ such that $\nu(q) = 0$, $q$ is invertible in $R_p$. If we had an isomorphism $\phi$ from $R_p$ to $\mathbb{Z}$, we would have $\phi(qq^{-1}) = \phi(q)\phi(q^{-1}) = 1$ and therefore $\phi(q) = \phi(q^{-1}) = 1$. Contradicting the bijectivity.