Theory and Methods for Reinforcement Learning

Prof. Volkan Cevher
volkan.cevher@epfl.ch

Lecture 4: Linear Programming

Laboratory for Information and Inference Systems (LIONS)
École Polytechnique Fédérale de Lausanne (EPFL)

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lions@epfl

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Recap - Reinforcement learning objective

- Reinforcement Learning: Sequential decision making in unknown environment
- Markov decision process: $M = (\mathcal{S}, \mathcal{A}, P, r, \mu, \gamma)$
- Stationary stochastic policy $\pi : \mathcal{S} \to \Delta(\mathcal{A})$, $a_t \sim \pi(\cdot|s_t)$
- State-value function: $V^\pi(s) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) | s_0 = s, \pi\right]$
- Performance objective: $\max_\pi (1-\gamma) \sum_{s \in \mathcal{S}} \mu(s) V^\pi(s)$

Challenges:
- Infer long-term consequences based on limited, noisy short-term feedback.
- Unknown dynamics - Knowledge only through sampled experience.
- Large state and actions spaces.
- Highly nonconvex objective.

Motivation

◦ Approximate dynamic programming

- Attempting to find approximate fixed-point solutions to the (nonlinear) Bellman equation.
- Pros:
  - + Well studied setting for tabular MDPs that comes with theoretical convergence guarantees.
    - See Lectures 2 and 3.
  - + Deep-learning variants (e.g., DQN [18]) are powerful.
- Cons:
  - – Training can oscillate or even diverge under the simplest parameterizations or in offline settings.
    - For divergent examples for TD-learning with nonlinear parameterizations, see e.g., Ex 6.6 and 6.7 in [3].
    - For divergent example for approximate VI with linear parameterizations, see e.g., Ex. 6.11 in [3].
  - – Incompatible with classical machine-learning tools that are rooted in convex optimization.

Motivation (cont.)

◦ The linear programming approach

- Introduces an alternative convex viewpoint that formulates the RL problem as a linear program.
- Overviews recent scalable algorithms with theoretical guarantees rooted in the LP approach.
- Highlights how historical key limitations have been eliminated.

Revisit Bellman optimality equation

- Finding $V^\star$ satisfying Bellman optimality equation can be written as a feasibility problem:

$$\min_{V} \;\; 0$$
$$\text{s.t.} \;\; V(s) = \max_{a\in\mathcal{A}} \left[r(s,a) + \gamma \sum_{s'\in\mathcal{S}} \mathsf{P}(s'|s,a)V(s') \right], \quad \forall\, s \in \mathcal{S}.$$

- The only feasibile point is $V^\star$.
- The above constraint suggests that $V^\star(\cdot)$ is the "least feasible solution" of $V(\cdot)$ satisfying

$$V(s) \geq \; r(s,a) + \gamma \sum_{s'\in\mathcal{S}} \mathsf{P}(s'|s,a)V(s'), \quad \forall\, s \in \mathcal{S},\; a \in \mathcal{A}.$$

- Note that the new constraints above is linear in $V(\cdot)$ $\quad\Longrightarrow\quad$ Linear Programming (LP) .

Solving MDPs with LP - Primal LP formulation

Primal LP

Let $\mu(s) > 0, s \in \mathcal{S}$ be the initial distribution (or any positive weights).

$$\begin{aligned} \min_{V} \quad & (1-\gamma) \sum_{s \in \mathcal{S}} \mu(s) V(s) \\ \text{s.t.} \quad & V(s) \geq \; r(s,a) + \gamma \sum_{s' \in \mathcal{S}} \mathsf{P}(s'|s,a) V(s'), \quad \forall\, s \in \mathcal{S},\; a \in \mathcal{A}. \end{aligned} \tag{P}$$

Remarks:

- The optimal value function $V^\star$ is the unique solution to the above LP.
- Number of decision variables: $|\mathcal{S}|$, number of constraints: $|\mathcal{S}| \times |\mathcal{A}|$.
- An optimal (deterministic) policy is the associated greedy policy

$$\pi^\star(s) \in \arg\max_{a \in \mathcal{A}} \left[r(s,a) + \gamma \sum_{s' \in \mathcal{S}} \mathsf{P}(s'|s,a) V^\star(s) \right]. \tag{1}$$

- The factor $(1-\gamma)$ in the objective ensures that the dual variables are in the probability simplex.

Solving MDPs with LP - Primal LP formulation (cont.)

Corollary (LP Formulation and $V^\star$)

$V^\star$ is the solution to the above LP formulation for any positive weights $\{\mu(s)\}$.

Proof Sketch

- *First, we establish that $V^\star$ is a feasible solution.*
- *Constraints hold for any $a \in \mathcal{A}$, so choose $a = \pi^\star(s)$, the deterministic stationary optimal policy.*
- *Hence, we have $\left(I - \gamma P^{\pi^\star}\right)V^\star \geq R^{\pi^\star}$.*
- *Then, we need to show that $V^\star$ minimizes the objective.*
- *By the monotonicity property of the Bellman operator, we get that $V \geq V^\star$, for any feasible V.*

Remark:

- When we introduce function approximations, the quality of the approximate minimizer V depends on the choice of the positive weights $\{\mu(s)\}$ (see Slide 19).

A closer look at the primal LP

Recall: Primal LP

Let $\mu(s) > 0, s \in \mathcal{S}$ be the initial distribution (or any positive weights).

$$\begin{aligned} \min_{V} \quad & (1-\gamma)\sum_{s\in\mathcal{S}} \mu(s)V(s) \\ \text{s.t.} \quad & V(s) \geq \; r(s,a) + \gamma \sum_{s'\in\mathcal{S}} \mathsf{P}(s'|s,a)V(s'), \quad \forall\, s \in \mathcal{S},\; a \in \mathcal{A}. \end{aligned} \tag{P}$$

Observations:

- Any $V^\star$ is feasible as

  $$V^\star(s) = \mathcal{T}V^\star(s) \geq r(s,a) + \gamma \sum_{s'\in\mathcal{S}} \mathsf{P}(s'|s,a)V^\star(s'),\ \forall a \in \mathcal{A}.$$

  This implies feasibility.

- For any feasible V, we have $V \geq \mathcal{T}V$. By monotonicity of the Bellman operator $\mathcal{T}$, we have

  $$V \geq \mathcal{T}V \geq \mathcal{T}^2 V \geq \cdots \geq \mathcal{T}^\infty V = V^\star.$$

  This implies optimality.

Solving MDPs with dual LP

Dual LP formulation

$$
\begin{aligned}
\max_{\lambda \geq 0} \quad & \sum_{s \in \mathcal{S}} \sum_{a \in \mathcal{A}} r(s,a) \lambda(s,a) \\
\text{s.t.} \quad & \sum_{a \in \mathcal{A}} \lambda(s,a) = (1-\gamma)\mu(s) + \gamma \sum_{s' \in \mathcal{S}, a' \in \mathcal{A}} \mathsf{P}(s|s',a') \lambda(s',a'), \quad \forall\, s \in \mathcal{S}.
\end{aligned} \tag{D}
$$

Remarks:

- The number of decision variables: $|\mathcal{S}| \times |\mathcal{A}|$.
- The number of constraints: $|\mathcal{S}| + |\mathcal{S}| \times |\mathcal{A}|$.
- The constraints implicitly implies the decision variables are in the probability simplex.
- The solution to the dual LP, $\lambda^\star$, corresponds to the state-action occupancy of $\pi^\star$.

Dual interpretation

- For any policy π and $s_0 \sim \mu$, define the state-action visitation distribution $\lambda^\pi(s,a)$ as

$$\lambda^\pi(s,a) := (1-\gamma)\sum_{t=0}^{\infty}\gamma^t \mathbb{P}(s_t = s, a_t = a \,|\, s_0 \sim \mu, \pi)$$

- We can write

$$(1-\gamma)\mathbb{E}_{s\sim\mu}[V^\pi(s)] = (1-\gamma)\,\mathbb{E}\left[\sum\nolimits_{t=0}^{\infty}\gamma^t r(s_t,a_t) \mid s_0 \sim \mu\right] \quad \Rightarrow \text{ primal objective (P)}$$

$$= (1-\gamma)\sum_{s\in\mathcal{S},a\in\mathcal{A}}\sum_{t=0}^{\infty}\gamma^t \mathbb{P}(s_t = s, a_t = a \,|\, s_0 \sim \mu, \pi) r(s,a)$$

$$= \sum_{s\in\mathcal{S}}\sum_{a\in\mathcal{A}}\lambda^\pi(s,a) r(s,a) \quad \Rightarrow \text{ dual objective (D)}$$

- Easy to verify that $\lambda^\pi(s,a)$ satisfies the constraints in the dual LP.

A closer look at the dual LP

Recall: Dual LP

$$\begin{aligned} \max_{\lambda} \quad & \sum_{s\in\mathcal{S}}\sum_{a\in\mathcal{A}} r(s,a)\lambda(s,a) \\ \text{s.t.} \quad & \sum_{a\in\mathcal{A}} \lambda(s,a) = (1-\gamma)\mu(s) + \gamma \sum_{s'\in\mathcal{S},a'\in\mathcal{A}} \mathsf{P}(s|s',a')\lambda(s',a'), \quad \forall\, s\in\mathcal{S}, \\ & \lambda(s,a)\geq 0, \quad \forall\, s\in\mathcal{S}, a\in\mathcal{A}. \end{aligned} \tag{D}$$

Observations:

- For any policy π and $s_0 \sim \mu$, define the state-action visitation distribution

$$\lambda^{\pi}(s,a) := (1-\gamma)\sum_{t=0}^{\infty}\gamma^t \mathbb{P}(s_t = s, a_t = a \,|\, \pi, s_0\sim\mu).$$

  By Markov property, we have (see supplementary material for details)

$$\lambda^{\pi}(s,a) = \mu(s)\pi(a|s) + \gamma\sum_{s',a'}\pi(a|s)\mathsf{P}(s|s',a')\lambda^{\pi}(s',a').$$

  Summing over a implies feasibility.

A closer look at the dual LP

Dual LP

$$\max_{\lambda} \sum_{s\in\mathcal{S}} \sum_{a\in\mathcal{A}} r(s,a)\lambda(s,a)$$
$$\text{s.t.} \sum_{a\in\mathcal{A}} \lambda(s,a) = (1-\gamma)\mu(s) + \gamma \sum_{s'\in\mathcal{S},a'\in\mathcal{A}} \mathsf{P}(s|s',a')\lambda(s',a'), \quad \forall\, s\in\mathcal{S},$$
$$\lambda(s,a) \geq 0, \quad \forall\, s\in\mathcal{S}, a\in\mathcal{A}. \tag{D}$$

Observation:

- We can show the objective equivalence of (P) and (D):

$$(1-\gamma)\sum_{s\in\mathcal{S}} \mu(s)V^{\pi}(s) = (1-\gamma)\mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t,a_t) \,|\, \pi, s_0\sim\mu\right]$$
$$\Updownarrow$$
$$\sum_{s\in\mathcal{S}} \sum_{a\in\mathcal{A}} r(s,a)\lambda^{\pi}(s,a) = (1-\gamma) \sum_{s\in\mathcal{S},a\in\mathcal{A}} \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t=s, a_t=a \,|\, \pi,\, s_0\sim\mu) r(s,a).$$

A closer look at the dual LP (cont.)

Dual LP

$$\begin{aligned} \max_{\lambda} \quad & \sum_{s\in\mathcal{S}}\sum_{a\in\mathcal{A}} r(s,a)\lambda(s,a) \\ \text{s.t.} \quad & \sum_{a\in\mathcal{A}} \lambda(s,a) = (1-\gamma)\mu(s) + \gamma \sum_{s'\in\mathcal{S},a'\in\mathcal{A}} \mathsf{P}(s|s',a')\lambda(s',a'), \quad \forall\, s\in\mathcal{S}, \\ & \lambda(s,a) \geq 0, \quad \forall\, s\in\mathcal{S}, a\in\mathcal{A}. \end{aligned} \tag{D}$$

Observations:

- For any λ feasible to the dual LP, we can define a policy

$$\pi_\lambda(a\,|\,s) = \frac{\lambda(s,a)}{\sum_{a\in\mathcal{A}}\lambda(s,a)}.$$

  It then holds $\lambda^{\pi_\lambda} = \lambda$.

- Note that $\lambda^\star(s,a) = \lambda^{\pi^\star}(s,a)$ and $\pi^\star(a\,|\,s) = \frac{\lambda^\star(s,a)}{\sum_{a\in\mathcal{A}}\lambda^\star(s,a)}$. (self-check)

- Optimal policy does not depend on μ. (LP sensitivity analysis)

Finding the optimal policy

- Primal LP approach:
  - Solve primal LP to obtain for the optimal value function $V^\star$
  - Then construct the optimal policy (deterministic) through the greedy policy

$$\pi^\star(s) \in \arg\max_{a \in \mathcal{A}} \left[r(s,a) + \gamma \sum\nolimits_{s' \in \mathcal{S}} \mathsf{P}(s'|s,a) V^\star(s') \right].$$

- Dual LP approach:
  - Solve the dual LP to obtain the optimal state-action occupancy $\lambda^\star$
  - Then construct the optimal policy (randomized) by

$$\pi^\star(a \,|\, s) = \frac{\lambda^\star(s,a)}{\sum_{a \in \mathcal{A}} \lambda^\star(s,a)}.$$

- Reference: [Puterman 1994] [25] (Section 6.9)

Dynamic programming vs Linear programming (exact solutions)

Algorithm	Component	Output
Value Iteration (VI)	Bellman Optimality Operator $\mathcal{T}$	$V^\star$ (control)
Policy Iteration (PI)	(Multiple) Bellman Operator $\mathcal{T}^\pi$ + Greedy Policy	$\pi^\star$ (control)
Linear Programming (LP)	LP solver (Simplex, Interior Point Method)	$V^\star, \pi^\star$ (control)

Dynamic Programming:

- Simple iterative updates.
- Polynomial complexity in $|\mathcal{S}|$ and $|\mathcal{A}|$.
- Works better for small problems.

Linear Programming:

- Rich library of fast LP solvers.
- Polynomial complexity in $|\mathcal{S}|$ and $|\mathcal{A}|$.
- Works better for large problems.

The LP approach - Pros and Cons

- Why is this useful?
  - Defining optimality is simple: no value functions, no fixed-point equations, just the numerical objective.
  - Easily comprehensible with an optimization background.
  - A disciplined convex optimization template with a rich set of algorithms.

- End User License Agreement:
  - Need to ensure $\sum_{a \in \mathcal{A}} \lambda(s, a) > 0$ to extract a policy.
  - Number of variables is large.
  - Intractable number of constraints.
  - Constraints may be not satisfied when working with function approximators.

Beyond exact solutions - A bit of history of approximate linear programming (ALP)

- [Manne 1960] [17]
  - Formulated the primal LP over value functions and showed equivalence to Bellman equations.
- [Borkar 1988] [4] and [Hérnandez-Lerma & Lasserre 1996, 1999] [11, 12]
  - Studied the LP approach to MDPs with continuous state and action spaces.
  - The corresponding LPs are infinite-dimensional.
- [Schweitzer & Seidman 1982] [29]
  - Proposed linear function approximators to reduce the number of decision variables
  - Proposed a relaxation to reduce the number of constraints.
- [De Farias & Van Roy 2003, 2004] [7, 8]
  - Analyzed the reduction [Schweitzer & Seidman 1982] [29].
  - Inspired some follow-up work in RL [Petrik et al. 2009,2010] [23, 22], [Desai et al. 2012] [9], [Abbasi-Yadkori et al. 2014] [1], [Lakshminarayanan et al. 2018] [15].

Prior works in ALP - Linear function approximation

Large-scale MDPs $\Rightarrow$ Large-scale optimization

- Reduce the number of decision variables by projecting onto a lower-dimensional subspace.
  - Let $\phi_1, \ldots, \phi_k : \mathcal{S} \to \mathbb{R}$ be k basis functions (or features).
  - $\Phi := \begin{bmatrix} \phi_1 & \ldots & \phi_k \end{bmatrix} \in \mathbb{R}^{|\mathcal{S}| \times k}$ is the corresponding feature matrix.
  - The (ALP) is obtained by adding the linear constraint $V = \Phi\theta = \sum_{i=1}^{k} \theta_i \phi_i$ to the original primal LP (P).

Approximate linear program [Schweitzer & Seidman 1982]

$$\begin{aligned} \min_{\theta \in \mathbb{R}^k} \quad & (1-\gamma) \sum_{s \in \mathcal{S}} \mu(s)(\Phi\theta)(s) \\ \text{s.t.} \quad & (\Phi\theta)(s) \geq \; r(s,a) + \gamma \sum_{s' \in \mathcal{S}} \mathrm{P}(s'|s,a)(\Phi\theta)(s'), \quad \forall \; s \in \mathcal{S}, \; a \in \mathcal{A}. \end{aligned} \tag{ALP}$$

Prior works in ALP - Linear function approximation (cont.)

Assumptions:

- The set $\{\phi_1, \ldots, \phi_k\}$ is linearly independent.
- $\mathbf{1} \in \text{span}\big(\{\phi_1, \ldots, \phi_k\}\big) := \{\Phi\theta \mid \theta \in \mathbb{R}^d\}$. This ensures that (ALP) is feasible [7] .
- The values $\sum_{s' \in \mathcal{S}} \mathsf{P}(s'|s,a)\phi_i(s')$ and $\mu^\top \phi_i$, $i = 1, \ldots, k$, can be accessed in $\mathcal{O}(1)$ time.

Quality of the approximate solution (Th.2 in [De Farias & Van Roy 2003] [7])

$$\|V^\star - V^\star_{\text{ALP}}\|_{1,\mu} \leq \frac{2}{1-\gamma} \underbrace{\min_\theta \|V^\star - \Phi\theta\|_\infty}_{\varepsilon_{\text{approx}}:\ \text{approximation error}}.$$

Notation:

- $\theta^\star_{\text{ALP}}$ is optimal to (ALP) and $V^\star_{\text{ALP}} = \Phi\theta^\star_{\text{ALP}}$ is the approximate value function.
- $\|V\|_{1,\mu} := \sum_{s \in \mathcal{S}} \mu(s)|V(s)|$ is the μ-weighted ℓ_1-norm, where $\mu > 0$.
- $\Phi\theta^\star$ is the $\|\cdot\|_\infty$-norm projection of $V^\star$ to the subspace $V = \Phi\theta$.
- $\varepsilon_{\text{approx}} := \min_\theta \|V^\star - \Phi\theta\|_\infty = \|V^\star - \Phi\theta^\star\|_\infty$ is called the approximation error.

Prior works in ALP - Linear function approximation (cont.)

Quality of the approximate solution

$$\|V^\star - V^\star_{\text{ALP}}\|_{1,\mu} \leq \frac{2}{1-\gamma}\varepsilon_{\text{approx}}.$$

Remarks:

- $\varepsilon_{\text{approx}} = \min_\theta \|V^\star - \Phi\theta\|_\infty$ captures the approximation power of the feature map.
- If $V^\star \in \text{span}\big(\phi_1, \ldots, \phi_k\big)$, then $V^\star = \Phi\theta^\star_{\text{ALP}}$.
- In general, $\|V^\star - V^\star_{\text{ALP}}\|_{1,\mu} = \mathcal{O}(\varepsilon_{\text{approx}})$.
- Focus on finding a good basis, leaving the search of the "right" weights to an LP solver.

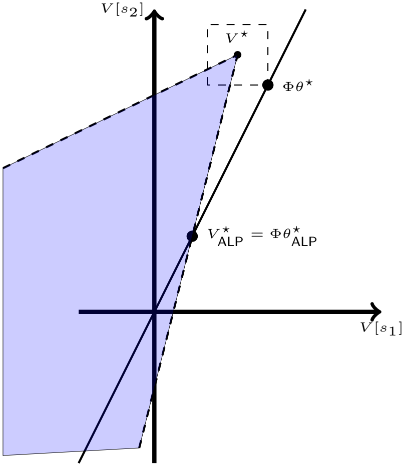

Figure: Graphical interpretation of ALP [7]

Prior works in ALP - Constraint sampling

◦ Reduce the number of constraints by constraint sampling.

- (x, a) is treated as an uncertainty parameter.
- $\mathcal{S} \times \mathcal{A}$ is the uncertainty space.
- $\mathbb{P}$ is a probability distribution on $\mathcal{S} \times \mathcal{A}$.
- $\{(s_i, a_i)\}_{i=1}^N$ i.i.d. samples on $(\mathcal{S} \times \mathcal{A}, \mathbb{P})$.
- $\mathcal{N} \subset \mathbb{R}^k$ is a bounding set.
- The relaxed LP (RLP) is obtained from (ALP) by considering only the N sampled constraints, and restricting $\theta \in \mathcal{N}$.

Relaxed linear program [De Farias & Van Roy 2001] [8]

$$\begin{aligned} \min_{\theta \in \mathcal{N}} \quad & (1-\gamma) \sum_{s \in \mathcal{S}} \mu(s)(\Phi\theta)(s) \\ \text{s.t.} \quad & (\Phi\theta)(s_i) \geq r(s_i, a_i) + \gamma \sum_{s' \in \mathcal{S}} \mathsf{P}(s'|s_i, a_i)(\Phi\theta)(s'), \quad \forall\, i = 1, \ldots, N. \end{aligned} \tag{RLP}$$

Prior works in ALP - Constraint sampling (cont.)

Assumptions:

- The set $\mathcal{N} \subset \mathbb{R}^k$ is compact, i.e., bounded and closed.
- The optimal solution $\theta^\star_{\text{ALP}}$ to (ALP) is in $\mathcal{N}$.
- The sampling probability distribution is $\mathbb{P} = \lambda^{\pi^\star}$, i.e., the state-action visitation distribution induced by an optimal policy $\pi^\star$.

How many samples give a good solution (Th.3.1 in [De Farias & Van Roy 2004] [8])

Let $\varepsilon, \delta \in (0,1)$. If $N \geq \tilde{\mathcal{O}}\Big(\frac{4k\log(\frac{1}{\delta})}{(1-\gamma)\varepsilon}\frac{\sup_{\theta\in\mathcal{N}}\|V^\star - \Phi\theta\|_\infty}{\mu^\top V^\star}\Big)$, then with probability at least $1-\delta$, we have

$$\|V^\star - V^\star_{\text{RLP}}\|_{1,\mu} \leq \|V^\star - V^\star_{\text{ALP}}\|_{1,\mu} + \varepsilon\|V^\star\|_{1,\mu},$$

where the probability is taken over the random sampling of constraints.

Notation:

- $\theta^\star_{\text{RLP}}$ is optimal to (RLP) and $V^\star_{\text{RLP}} = \Phi\theta^\star_{\text{RLP}}$ is the approximate value function.
- $\varepsilon \in (0,1)$ is the desired approximation accuracy.
- $\delta \in (0,1)$ is the desired confidence level.

Prior works in ALP - Constraint sampling (cont.)

Remarks:

- (RLP) is a relaxation of (ALP).
- The constraint $\theta \in \mathcal{N}$ ensures that that the optimal value of (RLP) is bounded.
- The relaxed linear program (RLP) is random.
- $\theta^\star_{\text{RLP}}$ and $V^\star_{\text{RLP}} = \Phi\theta^\star_{\text{RLP}}$ are random variables.
- A lower bound on the number of samples needed to achieve an ε-accurate solution with probability at least $1-\delta$, is called the sample complexity of the problem.
- The sample complexity bound depends on the choice of the bounding set $\mathcal{N}$.
- The sample complexity bound requires access to samples from the optimal state-action visitation distribution (which is not known a priori).

Common theme of all prior ALP works

- Reduce the number of decision variables by projecting on a low-dimensional subspace.
- Reduce the number of constraints (e.g., by constraint sampling).
- Solve the resulted LP with generic solver.
- Analyze the quality of the approximate solution.
- Either scale badly with the size of the state-action spaces or
- Require access to samples from a distribution that depends on the optimal policy.
- Require knowledge of dynamics or access to a simulator.
- Focus mainly on the approximation of the optimal value but not so much on the near optimal policy.

Is this the best we can do?

Some notation: towards an unconstrained problem.

- We will write an equivalent unconstrained problem.
- To simplify the notation we need to introduce a couple of operators:
  - $E : \mathbb{R}^{\mathcal{S}\times\mathcal{A}} \to \mathbb{R}^{\mathcal{S}}$ such that $(EV)(s,a) = V(s)$.
  - $P : \mathbb{R}^{\mathcal{S}\times\mathcal{A}} \to \mathbb{R}^{\mathcal{S}}$ such that $(PV)(s,a) = \sum_{s'} \mathsf{P}(s'|s,a)V(s')$.
- Their adjoint are:
  - $E^T : \mathbb{R}^{\mathcal{S}} \to \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ such that $(E^T\lambda)(s) = \sum_a \lambda(s,a)$.
  - $P^T : \mathbb{R}^{\mathcal{S}} \to \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ such that $(P^T\lambda)(s') = \sum_{s,a} \mathsf{P}(s'|s,a)\lambda(s,a)$.

Linear Programming - Summary

Primal LP:

$$\min_{V \in \mathbb{R}^{|\mathcal{S}|}} \quad (1-\gamma)\langle \mu, V \rangle \qquad \text{s.t.} \quad EV \geq r + \gamma P V. \tag{P}$$

Dual LP

$$\max_{\lambda \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}} \quad \langle \lambda, r \rangle \qquad \text{s.t.} \quad E^\top \lambda = (1-\gamma)\mu + \gamma P^\top \lambda, \quad \lambda \geq 0. \tag{D}$$

- Primal LP over value functions
- $|\mathcal{S}|$ decision variables and $|\mathcal{S}||\mathcal{A}|$ constraints
- $\forall\ V$ primal feasible $\Rightarrow V^\star \leq V$
- Optimal value function $V^\star$ is the optimizer
- Optimal policy is the associated greedy policy

- Dual LP over occupancy measures
- $|\mathcal{S}||\mathcal{A}|$ variables and $|\mathcal{S}| + |\mathcal{S}||\mathcal{A}|$ constraints
- $\forall$ policy π, the induced λ^π is dual feasible
- $\forall$ feasible $\lambda \Rightarrow \pi_\lambda$ has occupancy measure λ
- We have $\lambda^\star = \lambda^{\pi^\star}$ and $\pi^\star = \pi_{\lambda^\star}$

Towards the Lagrangian

- Instead of working solely with the primal or dual LP formulation, we work with an object between them
- Introducing the Lagrangian multipliers vector $\lambda \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$, we can write the Lagrangian as follows:

Primal LP:

$$\min_{V \in \mathbb{R}^{|\mathcal{S}|}} \quad (1-\gamma)\langle \mu, V\rangle \qquad \text{s.t.} \quad EV \geq r + \gamma PV. \tag{P}$$

Dual LP

$$\max_{\lambda \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}} \quad \langle \lambda, r\rangle \qquad \text{s.t.} \quad E^\top \lambda = (1-\gamma)\mu + \gamma P^\top \lambda, \quad \lambda \geq 0. \tag{D}$$

$$\Updownarrow$$

Saddle point formulation

$$\min_{V} \max_{\lambda \geq 0} (1-\gamma) \sum_{s \in \mathcal{S}} \langle \mu\,, V\rangle + \langle \lambda\,, r + \gamma PV + EV\rangle. \qquad \text{(Saddle-point problem)}$$

Minimax optimization

Bilinear min-max template

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle - h(\mathbf{y}),$$

where $\mathcal{X} \subseteq R^p$ and $\mathcal{Y} \subseteq \mathbb{R}^n$.

- $f: \mathcal{X} \to \mathbb{R}$ is convex.
- $h: \mathcal{Y} \to \mathbb{R}$ is convex.

Convex-concave min-max template

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y}), \tag{2}$$

where $\Phi(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}$ and concave in $\mathbf{y}$.

Basic algorithms for minimax

- Given $\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})$, define $V(\mathbf{z}) = [\nabla_{\mathbf{x}}\Phi(\mathbf{x},\mathbf{y}), -\nabla_{\mathbf{y}}\Phi(\mathbf{x},\mathbf{y})]$ with $\mathbf{z} = [\mathbf{x},\mathbf{y}]$.

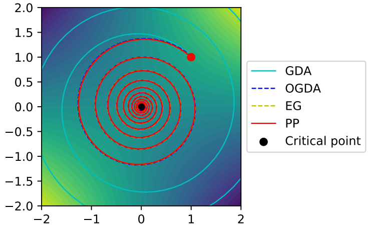

Figure: Trajectory of different algorithms for a simple bilinear game $\min_x \max_y xy$.

- (In)Famous algorithms
  - Gradient Descent Ascent (GDA)
  - Proximal point method (PPM)
  - Extra-gradient (EG)
  - Optimistic Gradient Descent Ascent (OGDA)
  - Reflected-Forward-Backward-Splitting (RFBS)

- EG and OGDA are approximations of the PPM
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \eta V(\mathbf{z}^k)$.
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \eta V(\mathbf{z}^{k+1})$.
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \eta V(\mathbf{z}^k - \alpha V(\mathbf{z}^{k-1}))$
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \eta [2V(\mathbf{z}^k) - V(\mathbf{z}^{k-1})]$
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \eta V(2\mathbf{z}^k - \mathbf{z}^{k-1})$

Primal-dual π-learning

Saddle point formulation

$$\min_{V} \max_{\lambda \in \Delta_{\mathcal{S}\times\mathcal{A}}} (1-\gamma)\sum_{s\in\mathcal{S}}\langle \mu\,, V\rangle + \langle \lambda\,, r + \gamma P V + E V\rangle. \qquad \text{(Saddle-point problem)}$$

- For know dynamic, it can be solved via primal-dual updates:
  - $V_{k+1} = V_k - \eta\Big((\gamma P - E)^\intercal \lambda_k + \mu\Big)$.
  - $\lambda_{k+1} = \lambda_k \odot e^{\eta(r+\gamma P V_k - E V_k)}$, where $\odot$ denotes entry wise multiplication.
- Gradients are expectations under μ_k and P

$\Rightarrow$ efficient stochastic implementation [Chen et al. 2018] [6], [Jin & Sidford. 2018] [13].

  - State-of-the art sample complexity for solving small MDPs.
  - $\mathcal{O}\left(\frac{|\mathcal{S}||\mathcal{A}|\log(\frac{1}{\delta})}{(1-\gamma)^4\varepsilon^2}\right)$ samples for finding an ε-optimal policy with probability at least $1-\delta$.

Scaling up

Large-scale MDPs $\Rightarrow$ Large-scale optimization

- Parameterize λ and V via linear functions
  - $\lambda_\nu = \Psi\nu$, for some feature matrix $\Psi \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|\times n}$
  - $V_\theta = \Phi\theta$, for some feature matrix $\Phi \in \mathbb{R}^{|\mathcal{S}|\times m}$

Relaxed saddle point formulation

$$\min_{\theta} \max_{\nu\in\Delta_{[n]}} (1-\gamma)\sum_{s\in\mathcal{S}}\langle\mu\,,\,\Phi\theta\rangle + \langle\lambda\,,\,\Psi^\top(r+\gamma P\Phi\theta + E\Phi\theta)\rangle$$

Scaling up (cont.)

Relaxed saddle point formulation

$$\min_{\theta} \max_{\nu \in \Delta_{[n]}} (1-\gamma) \sum_{s \in \mathcal{S}} \langle \mu \, , \, \Phi\theta \rangle + \langle \lambda \, , \, \Psi^\intercal (r + \gamma PV + EV) \rangle$$

◦ Primal-dual updates:

- $\theta_{k+1} = V\theta_k - \eta\Big((\gamma P\Phi - E\Phi)^\intercal \Psi \nu_k + \Phi^\intercal \mu\Big),$
- $\nu_{k+1} = \nu_k \odot e^{\eta \Psi^\intercal (r + \gamma P \Phi \theta_k - E\Phi\theta_k)}.$

◦ Implementable with only sample access to the columns of Ψ and the transition law P [Chen et al. 2018] [6].

- $\mathcal{O}\left(\frac{n\, m \log(\frac{1}{\delta})}{(1-\gamma)^4 \varepsilon^2}\right)$ samples for finding an $\varepsilon + \varepsilon_{\text{approx}}$-optimal policy with probability at least $1-\delta$.
- $\varepsilon_{\text{approx}}$ captures the expressivity of the approximation architecture.

Proximal point method (PPM)

◦ Consider the following smooth unconstrained optimization problem: $\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x})$

Proximal point method for convex minimization.

For a step-size $\tau > 0$, PPM can be written as follows

$$\mathbf{x}^{k+1} = \arg\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) + \frac{1}{2\tau}\|\mathbf{x} - \mathbf{x}^k\|^2 \right\} := \mathrm{prox}_{\tau f}(\mathbf{x}^k) \tag{3}$$

Observations: ◦ The optimality condition of (3) reveals a simpler PPM recursion for smooth f:

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \tau\nabla f(\mathbf{x}^{k+1}).$$

◦ PPM is an **implicit**, non-practical algorithm since we need the point $\mathbf{x}^{k+1}$ for its update.

◦ Each step of PPM can be as hard as solving the original problem.

◦ Convergence properties are well understood due to Rockafellar [28].

PPM and minimax optimization

PPM applied to the minimax template: $\min_{\mathbf{x}\in\mathbb{R}^d}\max_{\mathbf{y}\in\mathbb{R}^n}\Phi(\mathbf{x},\mathbf{y})$

Define $\mathbf{z} = [\mathbf{x},\mathbf{y}]^\top$ and $\mathbf{V}(\mathbf{z}) = [\nabla_{\mathbf{x}}\Phi(\mathbf{x},\mathbf{y}), -\nabla_{\mathbf{y}}\Phi(\mathbf{x},\mathbf{y})]^\top$. PPM iterations with a step-size $\tau > 0$ is given by

$$\mathbf{z}^{k+1} = \mathbf{z}^k - \tau\mathbf{V}(\mathbf{z}^{k+1}).$$

Derivation:

- For $\tau > 0$, $(\mathbf{x}^{k+1},\mathbf{y}^{k+1})$ is the unique solution to the saddle point problem,

$$\min_{\mathbf{x}\in\mathbb{R}^d}\max_{\mathbf{y}\in\mathbb{R}^n}\Phi(\mathbf{x},\mathbf{y}) + \frac{1}{2\tau}\|\mathbf{x}-\mathbf{x}^k\|^2 - \frac{1}{2\tau}\|\mathbf{y}-\mathbf{y}^k\|^2 \tag{4}$$

- Writing the optimality condition of the update in (4)

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \tau\nabla_{\mathbf{x}}\Phi(\mathbf{x}^{k+1},\mathbf{y}^{k+1}), \qquad \mathbf{y}^{k+1} = \mathbf{y}^k + \tau\nabla_{\mathbf{y}}\Phi(\mathbf{x}^{k+1},\mathbf{y}^{k+1}) \tag{5}$$

Observation:

- **PPM is an implicit algorithm.**
- For the bilinear problem, PPM is implementable!

Proximal point methods in the Bregman setup

Definition: Bregman distance

Let $\omega : \mathcal{X} \to \mathbb{R}$ be a distance generating function where ω is $1-$strongly convex w.r.t. some norm $\|\cdot\|$ on the underlying space and is continuously differentiable. The Bregman distance induced by $\omega(\cdot)$ is given by

$$D_\omega(\mathbf{z}, \mathbf{z}') = \omega(\mathbf{z}) - \omega(\mathbf{z}') - \nabla\omega(\mathbf{z}')^\top(\mathbf{z} - \mathbf{z}').$$

∘ The proximal point method in the Bregman setup reads as follows:

$$\boxed{\mathbf{x}^{k+1} = \arg\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) + \frac{1}{\tau} D_\omega(\mathbf{x}, \mathbf{x}^k) \right\}}$$

Remarks:

- ∘ Choosing the negative entropy as a generating function $\omega(\mathbf{x}) = \langle \mathbf{x}, \log \mathbf{x} \rangle$, we obtain the KL divergence. Such $\omega(\mathbf{x})$ is 1-strongly convex in $\|\cdot\|_1$ norm.
- ∘ This choice will allow to avoid projection in the simplex constraints and it improves the dependence on the domain dimension.
- ∘ Now, we will see PPM in action on the Lagrangian.

REPS: a success story

- REPS is widely popular in the robotics community.
- It is an application of proximal point to the Dual LP.
- A robot trained with REPS manages to play table tennis.

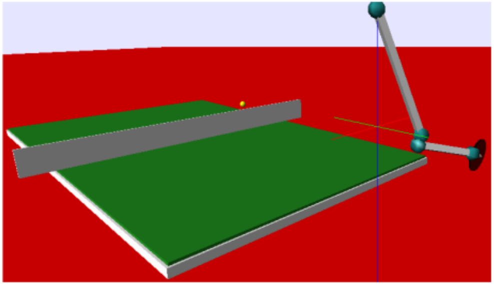

Figure: Source: Relative Entropy Policy Search [21]

Towards REPS: proximal point on the Dual LP

- Recall: Proximal point is an general implicit method.
- However, for a linear objective PPM can be implemented.
- Hence, we can apply proximal point updates on the Lagrangian that is just a bilinear form.

Recall: Dual LP

$$\begin{aligned} \lambda_k \;&=\; \mathrm{argmax}_{\lambda \in \Delta} \langle \lambda, r \rangle \\ &\qquad \text{s.t.} \quad E^T \lambda = \gamma P^T \lambda + (1-\gamma)\mu. \end{aligned}$$

Remarks: ◦ The problem in the current form suffers from $|\mathcal{S}|$ many constraints.

The Lagrangian: towards an unconstrained problem.

- The corresponding Lagrangian is:

$$\max_{\lambda\in\Delta}\min_{V}\ \langle\lambda, r\rangle + \langle V, \gamma P^T\lambda - E^T\lambda\rangle + (1-\gamma)\langle V, \mu\rangle.$$

- Applying proximal point we obtain the following update:

$$\lambda_k = \mathrm{argmax}_{\lambda\in\Delta} \underbrace{\min_{V}\ \langle\lambda, r\rangle + \langle V, \gamma P^T\lambda - E^T\lambda\rangle + (1-\gamma)\langle V, \mu\rangle}_{:=f(\lambda)} - \frac{1}{\eta} D_{KL}(\lambda, \lambda_{k-1}).$$

KKT conditions on the Lagrangian update.

Derivation:

- We notice by convexity of the Bregman divergence that the update is convex in λ.
- We introduce an auxiliary problem for any V as follows:

$$\lambda_k^V \;=\; \text{argmax}_{\lambda\in\Delta}\; \langle \lambda, r\rangle + \langle V, \gamma P^T\lambda - E^T\lambda\rangle + (1-\gamma)\langle V, \mu\rangle - \frac{1}{\eta} D_{KL}(\lambda, \lambda_{k-1}).$$

- By optimality conditions, it must hold

$$r + \gamma PV - EV - \frac{1}{\eta}\nabla_\lambda D_{KL}(\lambda_k^V, \lambda_{k-1}) = 0.$$

- Thus, λ_k^V can be computed in closed form for any V:

$$\lambda_k^V(s,a) = \frac{\lambda_{k-1}(s,a)e^{r(s,a)+\gamma(PV)(s,a)-(EV)(s,a)}}{\sum_{s,a}\lambda_{k-1}(s,a)e^{r(s,a)+\gamma(PV)(s,a)-(EV)(s,a)}}.$$

The unconstrained problem

◦ We can leverage the KKT conditions to write an unconstrained problem where the only decision variable is V:

$$\min_V \langle \lambda_k^V(s,a), r\rangle + \langle V, \gamma P^T \lambda_k^V(s,a) - E^T \lambda_k^V(s,a)\rangle + (1-\gamma)\langle V, \mu\rangle - \frac{1}{\eta} D_{KL}(\lambda_k^V(s,a), \lambda_{k-1}).$$

◦ With some calculus, we have the following compact form.

Unconstrained problem (REPS)

$$V_k = \min_V (1-\gamma)\langle \mu, V\rangle + \frac{1}{\eta}\log \sum_{s,a} \lambda_{k-1}(s,a) e^{r(s,a)+\gamma(PV)(s,a)-(EV)(s,a)}.$$

Remarks:

◦ The decision variable V has dimension $|\mathcal{S}|$.

◦ The objective is convex and smooth, its gradient is Lipschitz continuous.

The REPS algorithm [21]

Algorithm: REPS

Initialize λ_0, for exemple uniform.
for each iteration $k = 1, \ldots, K$ **do**
Solve the problem

$$V_k = \min_V (1-\gamma)\langle \mu, V \rangle + \frac{1}{\eta} \log \sum_{s,a} \lambda_{k-1}(s,a) e^{r(s,a) + \gamma (PV)(s,a) - (EV)(s,a)}$$

Update the occupancy measure:

$$\lambda_k(s,a) \propto \lambda_{k-1}(s,a) e^{r(s,a) + \gamma (PV_k)(s,a) - (EV_k)(s,a)}$$

end for

Sample complexity of REPS [20]

Algorithm	Oracle	Output
REPS	Exact gradient	$\mathcal{O}\left(\frac{\lvert\mathcal{S}\rvert^{3/2}}{(1-\gamma)^2\epsilon^2}\right)$
REPS	Stochastic Biased Gradients	$\mathcal{O}\left(\frac{\lvert\mathcal{S}\rvert^{3/2}}{(1-\gamma)^8\beta^2\epsilon^8}\right)$

Remarks:

- The exact gradient case achieves the best known sample complexity, e.g. comparable to NPG (see Lecture 6)
- The sample complexity with stochastic gradients degrades.
- For the stochastic gradient case, one needs to assume that $\lambda_k(s,a) \geq \beta > 0$. It solves the exploration problem by assumption.

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Supplementary

LP and optimization

Supplementary Material: Bellman Equation for State-action Visitation Distribution

Recall the definition

$$\lambda^\pi(s,a) := \sum_{t=0}^{\infty} \gamma^t P(s_t = s, a_t = a \,|\, \pi, s_0 \sim \mu).$$

Bellman Equation for λ^π

$$\lambda^\pi(s,a) = \mu(s)\pi(a|s) + \gamma \sum_{s',a'} \pi(a|s) P(s|s',a') \lambda^\pi(s',a').$$

Supplementary Material: Bellman Equation for State-action Visitation Distribution

Proof.

$$
\begin{aligned}
&\lambda^\pi(s,a)\\
=&P(s_0=s,a_0=a)+\sum_{t=1}^{\infty}\gamma^t P(s_t=s,a_t=a|\pi,s_0\sim\mu)\\
=&\mu(s)\pi(a|s)+\sum_{t=1}^{\infty}\gamma^t\sum_{s',a'}P(s_t=s,a_t=a|s_{t-1}=s',a_{t-1}=a',\pi,s_0\sim\mu)P(s_{t-1}=s',a_{t-1}=a'|\pi,s_0\sim\mu)\\
=&\mu(s)\pi(a|s)+\gamma\sum_{t=1}^{\infty}P(s_t=s,a_t=a|s_{t-1}=s',a_{t-1}=a')P(s_{t-1}=s',a_{t-1}=a'|\pi,s_0\sim\mu)\\
=&\mu(s)\pi(a|s)+\gamma\sum_{t=1}^{\infty}\pi(a|s)P(s|s',a')\sum_{t=1}^{\infty}\gamma^{t-1}P(s_{t-1}=s',a_{t-1}=a'|\pi,s_0\sim\mu)\\
=&\mu(s)\pi(a|s)+\gamma\sum_{s',a'}\pi(a|s)P(s|s',a')\lambda^\pi(s',a')
\end{aligned}
$$

where the third equality is due to Markov property. □

PPM guarantees for minimax optimization

Theorem (Convergence of PPM [28])

Suppose $(\mathbf{x}^k, \mathbf{y}^k)$ *be the iterates generated by PPM (i.e., (5)), then for the averaged iterates, it holds that*

$$\left| \Phi\left(\frac{1}{K}\sum_{k=1}^{K} \mathbf{x}^k, \frac{1}{K}\sum_{k=1}^{K} \mathbf{y}^k \right) - \Phi(\mathbf{x}^\star, \mathbf{y}^\star) \right| \le \frac{\|\mathbf{x}^0 - \mathbf{x}^\star\|^2 + \|\mathbf{y}^0 - \mathbf{y}^\star\|^2}{\tau K}.$$

Theorem (Linear convergence [28])

Suppose $(\mathbf{x}^k, \mathbf{y}^k)$ *be the iterates generated by (5),* $\Phi(\cdot, \cdot)$ *is* μ_x*−strongly convex in* $\mathbf{x}$ *and* μ_y*−strongly concave in* $\mathbf{y}$*. Let* $\mu = \max\{\mu_x, \mu_y\}$*. Then, for any* $\tau > 0$*,* $(\mathbf{x}^k, \mathbf{y}^k)$ *satisfies the following*

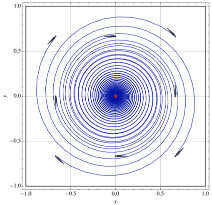
$$r^{k+1} \le \frac{1}{1 + \mu\tau} r^k,$$

where $r^k = \|\mathbf{x}^k - \mathbf{x}^\star\|^2 + \|\mathbf{y}^k - \mathbf{y}^\star\|^2$.

Remark:
- Still need an implementable and convergent algorithm beyond the stylized bilinear case.
- Note what happens when $\tau \to \infty$.

Extra-gradient algorithm (EG) [14]

EG method for saddle point problems
1. Choose $\mathbf{x}^0, \mathbf{y}^0$ and τ.
2. For $k = 0, 1, \cdots,$ perform:
$\tilde{\mathbf{x}}^k := \mathbf{x}^k - \tau \nabla_{\mathbf{x}} \Phi(\mathbf{x}^k, \mathbf{y}^k),$
$\tilde{\mathbf{y}}^k := \mathbf{y}^k + \tau \nabla_{\mathbf{y}} \Phi(\mathbf{x}^k, \mathbf{y}^k).$
$\mathbf{x}^{k+1} := \mathbf{x}^k - \tau \nabla_{\mathbf{x}} \Phi(\tilde{\mathbf{x}}^k, \tilde{\mathbf{y}}^k).$
$\mathbf{y}^{k+1} := \mathbf{y}^k + \tau \nabla_{\mathbf{y}} \Phi(\tilde{\mathbf{x}}^k, \tilde{\mathbf{y}}^k).$



◦ Idea: Predict the gradient at the next point

$$\mathbf{z}^{k+1} = \mathbf{z}^k - \tau \mathbf{V}\big(\underbrace{\mathbf{z}^k - \tau \mathbf{V}(\mathbf{z}^k)}_{\text{prediction of } \mathbf{z}^{k+1}} \big) \tag{EG}$$

Remark: ◦ 1-extra-gradient computation per iteration

Extra-gradient algorithm: Convergence

Theorem (General case [10])

Let $0 < \tau \leq \frac{1}{L}$. It holds that

- *Iterates $(\mathbf{x}^k, \mathbf{y}^k)$ remains bounded in a convex compact set.*
- *Primal-dual gap reduces:* $\mathrm{Gap}\left(\frac{1}{K}\sum_{k=1}^{K}\mathbf{x}^k, \frac{1}{K}\sum_{k=1}^{K}\mathbf{y}^k\right) \leq \mathcal{O}\left(\frac{1}{K}\right)$.

Theorem (Linear convergence [19])

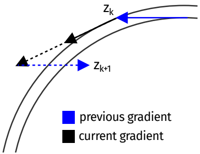
Suppose $(\mathbf{x}^k, \mathbf{y}^k)$ be the iterates generated by Extra-gradient algorithm, $\Phi(\cdot,\cdot)$ is μ_x−strongly convex in $\mathbf{x}$ and μ_y−strongly concave in $\mathbf{y}$. Let $\mu = \max\{\mu_x, \mu_y\}$. Then, for $\tau = \frac{1}{4L}$, $(\mathbf{x}^k, \mathbf{y}^k)$ satisfies,

$$r^{k+1} \leq \left(1 - \frac{1}{c\kappa}\right)^k r^0,$$

where $r^k = \|\mathbf{x}^k - \mathbf{x}^\star\|^2 + \|\mathbf{y}^k - \mathbf{y}^\star\|^2$, $\kappa = \frac{L}{\mu}$ is the condition number of the problem, and c is a constant which is independent of the problem parameters.

Optimistic gradient descent ascent algorithm (OGDA) [26]

OGDA for saddle point problems
1. Choose $\mathbf{x}^0, \mathbf{y}^0, \mathbf{x}^1, \mathbf{y}^1$ and τ.
2. For $k = 1, \cdots,$ perform:
$\mathbf{x}^{k+1} := \mathbf{x}^k - 2\tau\nabla_{\mathbf{x}}\Phi(\mathbf{x}^k, \mathbf{y}^k) + \tau\nabla_{\mathbf{x}}\Phi(\mathbf{x}^{k-1}, \mathbf{y}^{k-1})$.
$\mathbf{y}^{k+1} := \mathbf{y}^k + 2\tau\nabla_{\mathbf{y}}\Phi(\mathbf{x}^k, \mathbf{y}^k) - \tau\nabla_{\mathbf{y}}\Phi(\mathbf{x}^{k-1}, \mathbf{y}^{k-1})$.



◦ Main difference from the GDA: Add a "momentum" or "reflection" term to the updates

$$\mathbf{z}^{k+1} = \mathbf{z}^k - \tau\left[\mathbf{V}(\mathbf{z}^k) + \underbrace{(\mathbf{V}(\mathbf{z}^k) - \mathbf{V}(\mathbf{z}^{k-1}))}_{\text{momentum}}\right]. \qquad \text{(OGDA)}$$

◦ Known as Popov's method [24], it is also a special case of the Forward-Reflected-Backward method [16].

◦ It has ties to the Reflected-Forward-Backward Splitting (RFBS) method [5]:

$$\mathbf{z}^{k+1} = \mathbf{z}^k - \tau\mathbf{V}(2\mathbf{z}^k - \mathbf{z}^{k-1}). \qquad \text{(RFBS)}$$

Remark: ◦ Advanced material at the end: OGDA is an approximation of PPM for bilinear problems.

OGDA: Convergence

Theorem (General case [10])

Let $0 < \tau \leq \frac{1}{2L}$, $\mathbf{x}^1 = \mathbf{x}^0, \mathbf{y}^1 = y^0$. *It holds that*

- *Iterates* $(\mathbf{x}^k, \mathbf{y}^k)$ *remains bounded in a convex compact set.*
- *Primal-dual gap reduces:* $\mathrm{Gap}\left(\frac{1}{K}\sum_{k=1}^K \mathbf{x}^k, \frac{1}{K}\sum_{k=1}^K \mathbf{y}^k\right) \leq \mathcal{O}\left(\frac{1}{K}\right)$.

Theorem (Linear convergence [19])

Suppose $(\mathbf{x}^k, \mathbf{y}^k)$ *be the iterates generated by OGDA,* $\Phi(\cdot,\cdot)$ *is* μ_x-*strongly convex in* $\mathbf{x}$ *and* μ_y-*strongly concave in* $\mathbf{y}$. *Let* $\mu = \max\{\mu_x, \mu_y\}$. *Then, for* $\tau = \frac{1}{4L}$, $(\mathbf{x}^k, \mathbf{y}^k)$ *satisfies,*

$$r^{k+1} \leq \left(1 - \frac{1}{c\kappa}\right)^k r^0,$$

where $r^k = \|\mathbf{x}^k - \mathbf{x}^\star\|^2 + \|\mathbf{y}^k - \mathbf{y}^\star\|^2$, $\kappa = \frac{L}{\mu}$ *is the condition number of the problem, and* c *is a constant which is independent of the problem parameters.*

⋆Bregman divergences

Table: **Bregman functions** $\psi(\mathbf{x})$ & corresponding Bregman divergences/distances $d_\psi(\mathbf{x}, \mathbf{y})^a$.

Name (or Loss)	Domain[b]	$\psi(\mathbf{x})$	$d_\psi(\mathbf{x}, \mathbf{y})$
Squared loss	$\mathbb{R}$	x^2	$(x-y)^2$
Itakura-Saito divergence	$\mathbb{R}_{++}$	$-\log x$	$\frac{x}{y} - \log\left(\frac{x}{y}\right) - 1$
Squared Euclidean distance	$\mathbb{R}^p$	$\\|\mathbf{x}\\|_2^2$	$\\|\mathbf{x}-\mathbf{y}\\|_2^2$
Squared Mahalanobis distance	$\mathbb{R}^p$	$\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle$	$\langle (\mathbf{x}-\mathbf{y}), \mathbf{A}(\mathbf{x}-\mathbf{y})\rangle^c$
Entropy distance	p-simplex[d]	$\sum_i x_i \log x_i$	$\sum_i x_i \log\left(\frac{x_i}{y_i}\right)$
Generalized I-divergence	$\mathbb{R}_+^p$	$\sum_i x_i \log x_i$	$\sum_i \left(\log\left(\frac{x_i}{y_i}\right) - (x_i - y_i)\right)$
von Neumann divergence	$\mathbb{S}_+^{p\times p}$	$\mathbf{X}\log\mathbf{X} - \mathbf{X}$	$\mathrm{tr}\left(\mathbf{X}(\log\mathbf{X} - \log\mathbf{Y}) - \mathbf{X} + \mathbf{Y}\right)^e$
logdet divergence	$\mathbb{S}_+^{p\times p}$	$-\log\det\mathbf{X}$	$\mathrm{tr}\left(\mathbf{X}\mathbf{Y}^{-1}\right) - \log\det\left(\mathbf{X}\mathbf{Y}^{-1}\right) - p$

[a] $x, y \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$ and $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{p\times p}$.
[b] $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote non-negative and positive real numbers respectively.
[c] $\mathbf{A} \in \mathbb{S}_+^{p\times p}$, the set of symmetric positive semidefinite matrix.
[d] p-simplex:= $\{\mathbf{x} \in \mathbb{R}^p : \sum_{i=1}^p x_i = 1, x_i \geq 0, i = 1, \ldots, p\}$
[e] $\mathrm{tr}(\mathbf{A})$ is the trace of $\mathbf{A}$.

*Mirror descent [2]

What happens if we use a Bregman distance d_ψ in gradient descent?

Let $\psi : \mathbb{R}^p \to \mathbb{R}$ be a μ-strongly convex and continuously differentiable function and let the associated Bregman distance be $d_\psi(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x}) - \psi(\mathbf{y}) - \langle \mathbf{x} - \mathbf{y}, \nabla\psi(\mathbf{y})\rangle$.
Assume that the inverse mapping $\psi^\star$ of ψ is easily computable (i.e., its convex conjugate).

- **Majorize**: Find α_k such that

$$f(\mathbf{x}) \leq f(\mathbf{x}^k) + \langle \nabla f(\mathbf{x}^k), \mathbf{x} - \mathbf{x}^k\rangle + \frac{1}{\alpha_k} d_\psi(\mathbf{x}, \mathbf{x}^k) := Q^k_\psi(\mathbf{x}, \mathbf{x}^k)$$

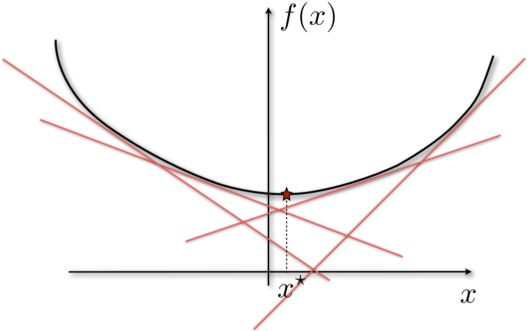
- **Minimize**

$$\begin{aligned}
\mathbf{x}^{k+1} &= \arg\min_{\mathbf{x}} Q^k_\psi(\mathbf{x}, \mathbf{x}^k) \Rightarrow \nabla f(\mathbf{x}^k) + \frac{1}{\alpha_k}\left(\nabla\psi(\mathbf{x}^{k+1}) - \nabla\psi(\mathbf{x}^k)\right) = 0 \\
\nabla\psi(\mathbf{x}^{k+1}) &= \nabla\psi(\mathbf{x}^k) - \alpha_k \nabla f(\mathbf{x}^k) \\
\mathbf{x}^{k+1} &= \nabla\psi^*(\nabla\psi(\mathbf{x}^k) - \alpha_k \nabla f(\mathbf{x}^k)) \qquad (\nabla\psi(\cdot))^{-1} = \nabla\psi^*(\cdot)[27].
\end{aligned}$$

- Mirror descent is a **generalization** of gradient descent for functions that are Lipschitz-gradient in norms other than the Euclidean.
- MD allows to deal with some **constraints** via a proper choice of ψ.

*What to keep in mind about mirror descent?

- Approximates the optimum by lower bounding the function via hyperplanes at $\mathbf{x}_t$



- The smaller the gradients, the better the approximation!

*Mirror descent example

How can we minimize a convex function over the unit simplex?

$$\min_{\mathbf{x}\in\Delta} f(\mathbf{x}),$$

where

- $\Delta := \{\mathbf{x} \in \mathbb{R}^p \ : \ \sum_{j=1}^p x_j = 1, \mathbf{x} \geq 0\}$ is the **unit simplex**;
- f is convex L_f-Lipschitz continuous with respect to some norm $\|\cdot\|$. (not necessarily *L-Lipschitz gradient*)

Entropy function

- Define the entropy function

$$\psi_e(\mathbf{x}) = \sum_{j=1}^{p} x_j \ln x_j \quad \text{if } \mathbf{x} \in \Delta, \quad +\infty \text{ otherwise.}$$

- ψ_e is 1-strongly convex over $\mathrm{int}\Delta$ with respect to $\|\cdot\|_1$.
- $\psi_e^\star(\mathbf{z}) = \ln \sum_{j=1}^p e^{z_j}$ and $\|\nabla\psi_e(\mathbf{x})\| \to \infty$ as $\mathbf{x} \to \tilde{\mathbf{x}} \in \Delta$.
- Let $\mathbf{x}^0 = p^{-1}\mathbf{1}$, then $d_\psi(\mathbf{x}, \mathbf{x}^0) \leq \ln p$ for all $\mathbf{x} \in \Delta$.

⋆Entropic descent algorithm [2]

Entropic descent algorithm (EDA)

Let $\mathbf{x}^0 = p^{-1}\mathbf{1}$ and generate the following sequence

$$x_j^{k+1} = \frac{x_j^k e^{-t_k f_j'(\mathbf{x}^k)}}{\sum_{j=1}^p x_j^k e^{-t_k f_j'(\mathbf{x}^k)}}, \quad t_k = \frac{\sqrt{2 \ln p}}{L_f} \frac{1}{\sqrt{k}},$$

where $f'(\mathbf{x}) = (f_1(\mathbf{x})', \ldots, f_p(\mathbf{x})')^T \in \partial f(\mathbf{x})$, which is the **subdifferential** of f at $\mathbf{x}$.

- This is an example of **non-smooth** and **constrained** optimization;
- The updates are multiplicative.

⋆Convergence of mirror descent

Problem

$$\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}) \tag{6}$$

where

- $\mathcal{X}$ *is a closed convex subset of* $\mathbb{R}^p$;
- f *is convex* L_f*-Lipschitz continuous with respect to some norm* $\|\cdot\|$.

Theorem ([2])

Let $\{\mathbf{x}^k\}$ *be the sequence generated by mirror descent with* $\mathbf{x}^0 \in \mathrm{int}\mathcal{X}$.
If the step-sizes are chosen as

$$\alpha_k = \frac{\sqrt{2\mu d_\psi(\mathbf{x}^\star, \mathbf{x}^0)}}{L_f}\frac{1}{\sqrt{k}}$$

the following convergence rate holds

$$\min_{0\leq s\leq k} f(\mathbf{x}^s) - f^\star \leq L_f\sqrt{\frac{2d_\psi(\mathbf{x}^\star, \mathbf{x}^0)}{\mu}}\frac{1}{\sqrt{k}}$$

- *This convergence rate is* **optimal** *for solving* (6) *with a first-order method.*