Exercice 1. (a) In both cases we have that $\{0\}$ and the whole ring are ideals. Therefore we will only search for non-trivial ideals.

We consider the quotient homomorphism

$$\xi : \mathbb{F}_3[t] \to \mathbb{F}_3[t] \big/ (t^2)$$

and we let $\overline{f} = \xi(f)$, where $f \in \mathbb{F}_3[t]$. Let I be a non-trivial ideal in $\mathbb{F}_3[t] \big/ (t^2)$. Then, by Proposition 2.4.38, there exists an ideal $J \subseteq \mathbb{F}_3[t]$ such that $(t^2) \subsetneq J \subsetneq \mathbb{F}_3[t]$, as I is non-trivial, and $\xi(J) = I$. As $\mathbb{F}_3$ is a field, we know that $\mathbb{F}_3[t]$ is principal and thus there exists $f(t) \in \mathbb{F}_3[t]$ such that $J = (f)$. As $t^2 \in (f)$, it follows that there exists $g(t) \in \mathbb{F}_3[t]$ such that $t^2 = f(t)g(t)$. Then we have that $\deg(f) \leq 2$. If $\deg(f) = 2$, respectively $\deg(f) = 0$, then $(f) = (t^2)$, respectively $(f) = \mathbb{F}_3[t]$, a contradiction. We deduce that $\deg(f) = 1$ and a quick calculation shows that $(f) = (t)$. We conclude that if I is a non-trivial ideal of $\mathbb{F}_3[t] \big/ (t^2)$, then $I = (\bar{t})$. Furthermore, as

$$(\mathbb{F}_3[t]/(t^2)) \big/ (\bar{t}) \cong \mathbb{F}_3[t] \big/ (t) \cong \mathbb{F}_3,$$

it follows that I is a maximal ideal in $\mathbb{F}_3[t] \big/ (t^2)$.

Similarly, we consider the quotient homomorphism

$$\xi : \mathbb{F}_2[t] \to \mathbb{F}_2[t] \big/ (t^3)$$

and we let $\overline{f} = \xi(f)$, where $f \in \mathbb{F}_2[t]$. Let I be a non-trivial ideal in $\mathbb{F}_2[t] \big/ (t^3)$. Then there exists an ideal $J \subseteq \mathbb{F}_2[t]$ with the property that $(t^3) \subsetneq J \subsetneq \mathbb{F}_2[t]$ and $\xi(J) = I$. As $\mathbb{F}_2[t]$ is principal, there exists $f \in \mathbb{F}_2[t]$ such that $J = (f)$. Now as $t^3 \in (f)$ it follows that $\deg(f) \leq 3$. As I is non-trivial, we deduce that $\deg(f) = 1$ or 2. If $\deg(f) = 1$, then $(f) = (t)$, while if $\deg(f) = 2$, then $(f) = (t^2)$. We conclude that if I is a non-trivial ideal of $\mathbb{F}_2[t] \big/ (t^3)$ then $I \in \{(\bar{t}), (\overline{t^2})\}$. Lastly, as

$$(\mathbb{F}_2[t] \big/ (t^3)) \big/ (\bar{t}) \cong \mathbb{F}_2[t] \big/ (t) \cong \mathbb{F}_2,$$

it follows that $(\bar{t})$ is a maximal ideal in $\mathbb{F}_2[t] \big/ (t^3)$. On the other hand $(\overline{t^2})$ is neither maximal, as $(\overline{t^2}) \subseteq (\bar{t})$, nor prime as $\bar{t} \cdot \bar{t} = \overline{t^2} \in (\overline{t^2})$ but $\bar{t} \notin (\overline{t^2})$.

(b) Let $I \subseteq M \subseteq A$ be two ideal in A. By Proposition 1.4.41 we have that:

$$A \big/ M \cong (A/I) \big/ \pi(M).$$

Now M is a maximal ideal in A if and only if $A \big/ M$ is a field. Now, by the above, $A \big/ M$ is a field if and only if $(A/I) \big/ \pi(M)$ is a field, hence if and only if $\pi(M)$ is a maximal ideal in $A \big/ I$.

Exercice 2. (a) Let $f(t), g(t) \in A[t]$. We have that

$$\mathrm{ev}(f+g)(a) = (f+g)(a) = f(a) + g(a) = \mathrm{ev}(f)(a) + \mathrm{ev}(g)(a) = (\mathrm{ev}(f) + \mathrm{ev}(g))(a)$$

for all $a \in A$. Therefore $\mathrm{ev}(f+g) = \mathrm{ev}(f) + \mathrm{ev}(g)$.

Similarly,

$$\mathrm{ev}(fg)(a) = (fg)(a) = f(a)g(a) = \mathrm{ev}(f)(a)\,\mathrm{ev}(g)(a) = (\mathrm{ev}(f)\,\mathrm{ev}(g))(a)$$

for all $a \in A$. Therefore $\mathrm{ev}(fg) = \mathrm{ev}(f)\,\mathrm{ev}(g)$.

Lastly, we have that $\mathrm{ev}(1)(a) = 1$ for all $a \in A$ and thus $\mathrm{ev}(1) = 1$, where the constant polynomial function 1 is the unity of $\mathcal{F}(A)$.

(b) Let $A = \mathbb{Z}/p\mathbb{Z}$ and let $f(t) = t^p - t \in A[t]$. Then $\mathrm{ev}(f)(a) = f(a) = a^p - a = 0$ for all $a \in A$ and thus $f \in \ker(\mathrm{ev})$.

(c) Let $A = \mathbb{R}$ and let $f(t) \in \ker(\mathrm{ev})$. Then, for all $a \in \mathbb{R}$ we have that $\mathrm{ev}(f)(a) = f(a) = 0$, which implies that all elements of $\mathbb{R}$ are roots of f. As f can have at most $\deg(f)$ real roots, we conclude that $f = 0$.

Exercice 3.
Let $\xi : F[x,y] \to A$ be the quotient homomorphism and for $f \in F[x,y]$ let $\overline{f} = \xi(f)$.

We first note that $(\overline{xy}) \subseteq \mathrm{nil}(A)$.

Let $(\overline{x}) \subseteq A$. By Exercise 3.1 of Series 4 we have that $\xi^{-1}(\overline{x}) = (x, x^2y^3) = (x)$, as $x^2y^3 \in (x)$ and by Proposition 1.4.41 we deduce that $(\overline{x})$ is a prime ideal of A as:

$$A/(\overline{x}) \cong F[x,y]\big/(x) \cong F[y] \quad \text{is an integral domain.}$$

Analogously one shows that $(\overline{y}) \subseteq A$ is a prime ideal.

We now consider the intersection $(\overline{x}) \cap (\overline{y})$. Let $\overline{f} \in (\overline{x}) \cap (\overline{y})$. Then, in particular, $\overline{f} \in (\overline{x})$ and so $\overline{f} = \overline{x} \cdot \overline{g}$, for some $\overline{g} \in A$. On the other hand, we have that $\overline{f} \in (\overline{y})$, hence $\overline{x} \cdot \overline{g} \in (\overline{y})$ and, as $(\overline{y})$ is prime, it follows that $\overline{g} \in (\overline{y})$. Therefore $\overline{f} = \overline{x} \cdot \overline{g} = \overline{x} \cdot \overline{y} \cdot \overline{h} = \overline{xy} \cdot \overline{h}$ for some $\overline{h} \in A$ and we deduce that $\overline{f} \in (\overline{xy})$. As the inclusion $(\overline{xy}) \subseteq (\overline{x}) \cap (\overline{y})$ is immediate, we determine that $(\overline{x}) \cap (\overline{y}) = (\overline{xy})$.

Finally we have that

$$\mathrm{nil}(A) = \bigcap_{\overline{p}-\text{prime}} \overline{p} \subseteq (\overline{x}) \cap (\overline{y}) = (\overline{xy})$$

and we conclude that $\mathrm{nil}(A) = (\overline{xy}) = (\overline{x}) \cap (\overline{y})$.

Exercice 4. (a) We first note that $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ is an $\mathbb{F}_p$-algebra: $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ is a commutative ring and $\psi : \mathbb{F}_p \to \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ given by $\psi(a) = a \cdot [0]$, for $a \in \mathbb{F}_p$, is a ring homomorphism with the property that $\psi(\mathbb{F}_p) \subseteq \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$. In particular, we have that $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ is an $\mathbb{F}_p$ vector space with basis $\{[0], [g], [2g], \ldots, [(p-1)g]\}$, where $[g]$ is a fixed generator of $\mathbb{Z}/p\mathbb{Z}$.

We now consider the evaluation homomorphism

$$\mathrm{ev}_{[g]} : \mathbb{F}_p[x] \to \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$$

$$\mathrm{ev}_{[g]}(x) = 1 \cdot [g].$$

We have that $(x^p - 1) \subseteq \ker(\mathrm{ev}_{[g]})$, as $\mathrm{ev}_{[g]}(x^p - 1) = 1 \cdot [pg] - 1 \cdot [0] = 0$. On the other hand, as $\mathbb{F}_p$ is a field, by Corollary 2.2.5, it follows that $\mathbb{F}_p[x]$ is a principal ring and thus there exists

$f \in \mathbb{F}_p[x]$ such that $\ker(\mathrm{ev}_{[g]}) = (f)$. Therefore, as $x^p - 1 \in (f)$, it follows that $x^p - 1 = f \cdot g$ for some $g \in \mathbb{F}_p[x]$ and by Lemma 2.1.1 we deduce that $\deg(f) \leq p$.

We write $f(x) = \sum_{i=0}^{p} a_i x^i$, where $a_i \in \mathbb{F}_p$. Then:

$$\mathrm{ev}_{[g]}(f(x)) = \sum_{i=0}^{m} a_i \cdot [ig] = (a_0 + a_p) \cdot [0] + \sum_{i=1}^{p-1} a_i \cdot [ig] = 0$$

and, as $[0], [g], [2g], \ldots, [(p-1)g]$ are linearly independent, we have $a_0 = -a_p$ and $a_i = 0$ for all $1 \leq i \leq p-1$. We deduce that $f(x) = a_p(x^p - 1)$, where $a_p \in \mathbb{F}_p$, and thus $\ker(\mathrm{ev}_{[g]}) = (x^p - 1)$.

In conclusion, we have shown that $\mathbb{F}_p[x]/(x^p - 1) \cong \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$.

(b) Recall that the characteristic is the natural number n such that $n\mathbb{Z}$ is the kernel of the unique ring homomorphism from $\mathbb{Z}$ to $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$. Note the unique ring homomorphism from $\mathbb{Z}$ to $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ sends $x \in \mathbb{Z}$ to $[x]_p \in \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$. Its kernel is $p\mathbb{Z}$ therefore $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ has characteristic p.

(c) Let $a = \sum_{i=0}^{p-1} a_i^{'} \cdot ([g] - 1)^i$ be an idempotent element of $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$. Then

$$a^2 = \sum_{i,j} a_i a_j \cdot [(i+j)g] = \sum_{k=0}^{p-1} a_k \cdot [kg] = a$$

and, as $[0], [g], \ldots, [(p-1)g]$ are linearly independent, it follows that $a_k = \sum_{i+j=k} a_i a_j$ for all $0 \leq k \leq p-1$. In particular, we have $a_0 = a_0^2$ and so $a_0 = 0$ or $a_0 = 1$. As $a_1 = a_0 a_1 + a_1 a_0$ we see that in both cases we obtain $a_1 = 0$. Recursively, we deduce that

$$a_{k+1} = \sum_{i+j=k+1} a_i a_j = a_0 a_{k+1} + \Big(\sum_{\substack{i+j=k+1 \\ 1 \leq i,j \leq k}} a_i a_j \Big) + a_0 a_{k+1} = a_0 a_{k+1} + a_{k+1} a_0$$

and therefore $a_{k+1} = 0$. Hence, if $a_0 = 0$, it follows that $a = 0 \cdot [0]$, while, if $a_0 = 1$, it follows that $a = 1 \cdot [0]$. We have shown that the only idempotents of $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ are $0 \cdot [0]$ and $1 \cdot [0]$. By Proposition 2.4.55 and Remark 2.4.56 we conclude that $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ cannot be decomposed as a product of non-zero rings.

Exercice 5.

We define $N : \mathbb{Z}[\sqrt{2}i] \to \mathbb{N} \cup \{-\infty\}$ by $N(x + \sqrt{2}iy) = x^2 + 2y^2$. We will show that $(\mathbb{Z}[\sqrt{2}i], N)$ is an Euclidean domain.

Let $0 \neq a \in \mathbb{Z}[\sqrt{2}i]$, where $a = x + \sqrt{2}iy$. Then $N(a) = x^2 + 2y^2 \in \mathbb{N}$.

Now let $a, b \in \mathbb{Z}[\sqrt{2}i]$, where $a \neq 0$. We want to show that there exist $q, r \in \mathbb{Z}[\sqrt{2}i]$ such that

$$b = qa + r \text{ and } N(r) < N(a).$$

Consider the complex number $\frac{b}{a}$. There exist $s, t \in \mathbb{R}$ such that $\frac{b}{a} = s + t\sqrt{2}i$. Let $x, y \in \mathbb{Z}$ be such that $|s - x| \leq \frac{1}{2}$ and $|t - y| \leq \frac{1}{2}$. Then $x + y\sqrt{2}i \in \mathbb{Z}[\sqrt{2}i]$ and

$$N((s + t\sqrt{2}i) - (x + y\sqrt{2}i)) \leq \left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 = \frac{3}{4}.$$

Let $q = x + y\sqrt{2}i$ and let $r = b - aq$. Then $r = \frac{r}{a} \cdot a = \left(\frac{b}{a} - q\right)a$ and

$$N(r) = N\left(\frac{b}{a} - q\right)N(a) \leq \frac{3}{4}N(a) < N(a).$$

1 Supplementary exercise

Exercice 6. (a) Let $\sum_{g\in G} a_g \cdot g \in \mathrm{Z}(A)$ and let $h \in G$. Then $1 \cdot h \in A$ is invertible with inverse $(1\cdot h)^{-1} = 1 \cdot h^{-1}$ and we have

$$(1\cdot h)(\sum_{g\in G} a_g \cdot g)(1\cdot h)^{-1} = \sum_{g\in G} a_g \cdot hgh^{-1} = \sum_{g'\in G} a_{h^{-1}g'h} \cdot g' = \sum_{g'\in G} a_{g'} \cdot g'.$$

It follows that $a_{h^{-1}gh} = a_g$ for all $h \in G$ and thus the map $g \to a_g$ is constant over equivalence classes.

Conversely, assume that $g \to a_g$ is constant over equivalence classes. Let $1 \cdot h \in A$. Then:

$$(1\cdot h)(\sum_{g\in G} a_g \cdot g)(1\cdot h)^{-1} = \sum_{g'\in G} a_{h^{-1}g'h} \cdot g' = \sum_{g'\in G} a_{g'} \cdot g'$$

and thus

$$(1\cdot h)(\sum_{g\in G} a_g \cdot g) = (\sum_{g\in G} a_g \cdot g)(1\cdot h), \text{ for all } h \in G.$$

Therefore

$$(\sum_{h\in G} a_h \cdot h)(\sum_{g\in G} a_g \cdot g) = \sum_{h\in G} a_h \cdot h \sum_{g\in G} a_g \cdot g = \sum_{h\in G} a_h (\sum_{g\in G} a_g g) h = (\sum_{g\in G} a_g \cdot g)(\sum_{h\in G} a_h \cdot h)$$

and consequently $\sum_{g\in G} a_g \cdot g \in \mathrm{Z}(A)$.

(b) Fix $A = \mathbb{C}[S_3]$. By (a) we have that $e_1, e_2, e_3 \in \mathrm{Z}(A)$. We will now show that they are idempotents. First,

$$\begin{aligned}
e_1^2 &= \frac{1}{36}(\sum_{g\in S_3} g)(\sum_{h\in S_3} h) \\
&= \frac{1}{36}\Big[\sum_{g\in S_3} g + \sum_{g\in S_3} g(12) + \sum_{g\in S_3} g(13) + \sum_{g\in S_3} g(23) + \sum_{g\in S_3} g(123) + \sum_{g\in S_3} g(132)\Big] \\
&= \frac{1}{6}\sum_{g\in S_3} g = e_1.
\end{aligned}$$

In the above we have used the fact that for all $x \in S_3$, the map $S_3 \to S_3$ sending $a \to ax$ is bijective. Hence $\sum_{g\in S_3} gx = \sum_{g\in S_3} g$ for all $x \in S_3$. Secondly,

$$\begin{aligned}
e_2^2 &= \frac{1}{36}(\sum_{g\in S_3} \mathrm{sgn}(g)g)(\sum_{h\in S_3} \mathrm{sgn}(h)h) \\
&= \frac{1}{36}\Big[\sum_{g\in S_3} \mathrm{sgn}(g)g - \sum_{g\in S_3} \mathrm{sgn}(g)g(12) - \sum_{g\in S_3} \mathrm{sgn}(g)g(13) - \sum_{g\in S_3} \mathrm{sgn}(g)g(23) + \\
&\quad + \sum_{g\in S_3} \mathrm{sgn}(g)g(123) + \sum_{g\in S_3} \mathrm{sgn}(g)g(132)\Big] \\
&= \frac{1}{36}\Big[\sum_{g\in S_3} \mathrm{sgn}(g)g - \sum_{g\in S_3} \mathrm{sgn}(g(12))g - \sum_{g\in S_3} \mathrm{sgn}(g(13))g - \sum_{g\in S_3} \mathrm{sgn}(g(23))g + \\
&\quad + \sum_{g\in S_3} \mathrm{sgn}(g(132))g + \sum_{g\in S_3} \mathrm{sgn}(g(123))g\Big] \\
&= \frac{1}{6}\sum_{g\in S_3} \mathrm{sgn}(g)g = e_2.
\end{aligned}$$

In the above we have used the fact that $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$ for all $\sigma, \tau \in S_3$.

Lastly, we will show that f_1 and f_2 are idempotents and that $f_1 f_2 = f_2 f_1 = 0$. We have that:

$$\begin{aligned} f_1^2 &= \frac{1}{9}\Big[\operatorname{Id} + \varepsilon(123) + \varepsilon^2(132) + \varepsilon(123) + \varepsilon^2(132) + \operatorname{Id} + \varepsilon^2(132) + \operatorname{Id} + \varepsilon(123)\Big] \\ &= \frac{1}{3}\Big[\operatorname{Id} + \varepsilon(123) + \varepsilon^2(132)\Big] = f_1. \end{aligned}$$

Analogously one shows that $f_2^2 = f_2$. Keeping in mind that $\varepsilon^2 + \varepsilon = -1$, we have

$$\begin{aligned} f_1 f_2 &= \frac{1}{9}\Big[\operatorname{Id} + \varepsilon(123) + \varepsilon^2(132) + \varepsilon^2(123) + (132) + \varepsilon \operatorname{Id} + \varepsilon(132) + \varepsilon^2 \operatorname{Id} + (123)\Big] \\ &= \frac{1}{9}(1 + \varepsilon + \varepsilon^2)\Big[\operatorname{Id} + (123) + (132)\Big] = 0. \end{aligned}$$

Analogously one shows that $f_2 f_1 = 0$. Therefore $e_3^2 = (f_1 + f_2)^2 = f_1^2 + f_1 f_2 + f_2 f_1 + f_2^2 = f_1 + f_2 = e_3$.

We have shown that e_1, e_2, e_3 are central idempotents. We will now show that they are pairwise orthogonal. We have

$$e_1 e_2 = \frac{1}{36}\Big[\sum_{g\in G} g - \sum_{g\in G} g(12) - \sum_{g\in G} g(13) - \sum_{g\in G} g(23) + \sum_{g\in G} g(123) + \sum_{g\in G} g(132)\Big] = 0.$$

Analogously one shows that $e_2 e_1 = 0$. We note that $e_3 = \frac{1}{3}(2\operatorname{Id} - (123) - (132))$. Then

$$e_1 e_3 = \frac{1}{18}\Big[2\sum_{g\in G} g - \sum_{g\in G} g(123) - \sum_{g\in G} g(132)\Big] = 0$$

and

$$\begin{aligned} e_2 e_3 &= \frac{1}{18}\Big[2\sum_{g\in G} \operatorname{sgn}(g)g - \sum_{g\in G} \operatorname{sgn}(g)g(123) - \sum_{g\in G} \operatorname{sgn}(g)g(132)\Big] \\ &= \frac{1}{18}\Big[2\sum_{g\in G} \operatorname{sgn}(g)g - \sum_{h\in G} \operatorname{sgn}(h(132))h - \sum_{h\in G} \operatorname{sgn}(h(123))h\Big] \\ &= 0. \end{aligned}$$

Now $e_1 + e_2 = \frac{1}{3}[\operatorname{Id} + (123) + (132)]$ is a central idempotent in A, as $(e_1 + e_2)^2 = e_1^2 + e_1 e_2 + e_2 e_1 + e_2^2 = e_1 + e_2$. Furthermore, one checks that $(e_1 + e_2)e_3 = 0$ and $e_1 + e_2 + e_3 = \operatorname{Id}$. Thus, by Proposition 1.4.55, we have that $A \cong A(e_1 + e_2) \times Ae_3$.

Similarly, e_1 and e_2 are central orthogonal idempotents in $A(e_1 + e_2)$ and, as $e_1 + e_2$ is the identity in $A(e_1 + e_2)$, we once more apply Proposition 1.4.55 to obtain $A(e_1 + e_2) \cong Ae_1 \times Ae_2$. We have shown that:

$$A \cong Ae_1 \times Ae_2 \times Ae_3.$$

(c) Let $x \in Ae_1$. Then $x = ye_1$, where $y = a_0 \operatorname{Id} + a_1(12) + a_2(13) + a_3(23) + a_4(123) + a_5(132) \in A$. We compute

$$\begin{aligned} x &= a_0 \sum_{g\in G} g + a_1 \sum_{g\in G} g(12) + a_2 \sum_{g\in G} g(13) + a_3 \sum_{g\in G} g(23) + a_4 \sum_{g\in G} g(123) + a_5 \sum_{g\in G} g(132) \\ &= (a_0 + a_1 + a_2 + a_3 + a_4 + a_5) \sum_{g\in G} g \\ &= (\sum_{i=0}^{5} a_i) e_1. \end{aligned}$$

Therefore if $x \in Ae_1$ then $x = c_x e_1$, for some $c_x \in \mathbb{C}$. Analogously, one shows that if $x \in Ae_2$ then $x = c_x e_2$, for some $c_x \in \mathbb{C}$. (In this case, computations will show that $c_x = a_0 - a_1 - a_2 - a_3 + a_4 + a_5$.)

For $i = 1, 2$ consider the map $\varphi : Ae_i \to \mathbb{C}$ given by $\varphi(x) = c_x$. One checks that φ is a ring isomorphism and concludes that $Ae_i \cong \mathbb{C}$, for $i = 1, 2$.

(d) Let $x \in Ae_3$. Then $x = ye_3$, where $y = a_0 \operatorname{Id} + a_1(12) + a_2(13) + a_3(23) + a_4(123) + a_5(132) \in A$. We compute

$$yf_1 = (a_0 + a_5\varepsilon + a_4\varepsilon^2)f_1 + (a_1 + a_2\varepsilon + a_3\varepsilon^2)(12)f_1$$

and

$$yf_2 = (a_0 + a_4\varepsilon + a_5\varepsilon^2)f_2 + (a_1 + a_3\varepsilon + a_2\varepsilon^2)(12)f_2$$

to determine that

$$\begin{aligned} x &= (a_0 + a_5\varepsilon + a_4\varepsilon^2)f_1 + (a_1 + a_2\varepsilon + a_3\varepsilon^2)(12)f_1 + (a_0 + a_4\varepsilon + a_5\varepsilon^2)f_2 + (a_1 + a_3\varepsilon + a_2\varepsilon^2)(12)f_2 \\ &= x_1 f_1 + x_2(12)f_1 + x_3(12)f_2 + x_4 f_2, \end{aligned}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{C}$.

Define the map $\varphi : Ae_3 \to M_2(\mathbb{C})$ by $\varphi(x) = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$. Clearly φ is a bijective map, $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in Ae_3$ and $\varphi(e_3) = \mathrm{I}_2$. What remains to show is that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in Ae_3$.

We first remark that

$$(12)f_1 = \frac{1}{3}[(12) + \varepsilon(23) + \varepsilon^2(13)] = f_2(12)$$

and

$$f_1(12) = \frac{1}{3}[(12) + \varepsilon(13) + \varepsilon^2(23)] = (12)f_2.$$

Now, keeping in mind that $f_1^2 = f_1$, $f_2^2 = f_2$, $f_1 f_2 = f_2 f_1 = 0$, $(12)f_1 = f_2(12)$ and $f_1(12) = (12)f_2$, we have

$$\begin{aligned} xy &= (x_1 f_1 + x_2(12)f_1 + x_3(12)f_2 + x_4 f_2)(y_1 f_1 + y_2(12)f_1 + y_3(12)f_2 + y_4 f_2) \\ &= x_1 y_1 f_1^2 + x_2 y_1 (12)f_1 f_1 + x_3 y_1 (12)f_2 f_1 + x_4 y_1 f_2 f_1 + x_1 y_2 f_1(12)f_1 + x_2 y_2 (12)f_1(12)f_1 + \\ &+ x_3 y_2 (12)f_2(12)f_1 + x_4 y_2 f_2(12)f_1 + x_1 y_3 f_1(12)f_2 + x_2 y_3 (12)f_1(12)f_2 + x_3 y_3 (12)f_2(12)f_2 + \\ &+ x_4 y_3 f_2(12)f_2 + x_1 y_4 f_1 f_2 + x_2 y_4 (12)f_1 f_2 + x_3 y_4 (12)f_2 f_2 + x_4 y_4 f_2^2 \\ &= x_1 y_1 f_1 + x_2 y_1 (12)f_1 + x_3 y_2 f_1 + x_4 y_2 (12)f_1 + x_1 y_3 (12)f_2 + x_2 y_3 f_2 + x_3 y_4 (12)f_2 + x_4 y_4 f_2 \\ &= (x_1 y_1 + x_3 y_2)f_1 + (x_2 y_1 + x_4 y_2)(12)f_1 + (x_1 y_3 + x_3 y_4)(12)f_2 + (x_2 y_3 + x_4 y_4)f_2. \end{aligned}$$

Thus $\varphi(xy) = \begin{pmatrix} x_1 y_1 + x_3 y_2 & x_1 y_3 + x_3 y_4 \\ x_2 y_1 + x_4 y_2 & x_2 y_3 + x_4 y_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 & y_3 \\ y_2 & y_4 \end{pmatrix} = \varphi(x)\varphi(y)$. We conclude that φ is a ring isomorphism and thus $Ae_3 \cong M_2(\mathbb{C})$.