Exercice 7. 1. Suppose the opposite, and let $c_1, \ldots, c_n$ be finitely many generators with $c_i = a_i/b_i$ for $a_i, b_i \in \mathbb{Z}$. Let $p_1, \ldots, p_s$ be the list of the distinct prime factors of the b_i's different from p.

Then, the following is a proper subring of $\mathbb{Z}_{(p)}$:

$$A = \{a/b \in \mathbb{Q} | b \text{ only has } p_1, \ldots, p_s \text{ as prime factors } \},$$

It is easy to check that A is stable under addition and multiplication and that the neutral and zero element are in A. Moreover, let q be a prime number different from p and $q \neq p_1, \ldots, p_s$. Then for any $a \in \mathbb{Z}$ we have that $a/q \in \mathbb{Z}_{(p)}$.

Hence the subring of $\mathbb{Z}_{(p)}$ generated by $c_1, \ldots, c_n$ is contained in $A \neq \mathbb{Z}_{(p)}$. This is a contradiction.

2. We will show that $\mathbb{Z}_p$ is generated by $1/p$ or with other words $\mathbb{Z}_p = \mathbb{Z}[1/p]$. In particular $\mathbb{Z}_p$ is a finitely generated ring.

   Note that every element in $\mathbb{Z}_p$ can be writen in the form $a/p^i = a \cdot (1/p) \cdots (1/p)$ with i factors of the form $1/p$. This means exactly that the subring $\mathbb{Z}[1/p]$, the subring generated by $1/p$, is equal to $\mathbb{Z}_p$. We can conclude that $\mathbb{Z}_p$ is finitely generated.

3. Indeed, $\mathbb{Z}$ is a subring of $\mathbb{Z}_p$. Now, let A be a subring of $\mathbb{Z}_p$ which is not equal to $\mathbb{Z}$. Since it contains the unit element, then it contains $\mathbb{Z}$, and at least one other element. This element, since it is not in $\mathbb{Z}$, is then necessarily of the form a/p^i where $i > 0$ and a is prime to p. By the Euclidean algorithm we have integers n and m such that $na + mp^i = 1$. Dividing by p^i yields $n(a/p^i) + m = 1/p^i$. Hence $1/p^i \in A$. If $i = 1$, then we obtain that $1/p \in A$. If $i > 1$, then by taking $(1/p^i) + ... + (1/p^i)$ with p^{i-1} summands we again obtain that $1/p \in A$. Then, by the previous point, since $1/p$ generates $\mathbb{Z}_p$, we have that $A = \mathbb{Z}_p$.

4. There cannot be a ring isomorphism since $\mathbb{Z}[1/p, 1/q]$ is generated by $1/p$ and $1/q$, while $\mathbb{Z}_{(p)}$ is not finitely generated.

5. We will show that we need only one 1 element to generate $\mathbb{Z}[1/p, 1/q]$. Note that $1/q$ and $1/p$ are a set of generators. One has that $1/p = 1/(pq) + \cdots + 1/(pq)$, with q-summands. We have the same for $1/q$, $1/p = 1/(pq) + \cdots + 1/(pq)$, with p-summands. Therefore we can conclude that $\mathbb{Z}[1/p, 1/q] = \mathbb{Z}[1/(pq)]$.