Exercice 8. 1. By Gauss lemma III $ax^2 + bx + c \in A[x]$ is not irreducible if and only if $ax^2 + bx + c \in K[x]$ is not irreducible. So,we show the equivalence of the right side of the statement to the latter.

Note now that $ax^2 + bx + c = a((x + b/2a)^2 - (b^2 - 4ac)/(4a^2))$, so $ax^2 + bx + c$ is irreducible in $K[x]$ if and only if $(x + b/2a)^2 - (b^2 - 4ac)/(4a^2) \in K[x]$ is irreducible if and only if $t^2 - (b^2 - 4ac)/(4a^2) \in K[t]$ is irreducible (by doing linear variable substitution $t = x + b/2a$, or with other words by applying the isomorphism $K[x] \cong K[t]$ induced by $t = x + b/2a$)

However, we have seen in the course that (as we are talking about a polynomial of degree 2), $t^2 - (b^2 - 4ac)/(4a^2) \in K[t]$ is not irreducible if and only if there is $u = f/g \in K$ (with $f, g \in Aandg \neq 0$) such that $u^2 = (b^2 - 4ac)/(4a^2) \in K$.

This latter is equivalent to the condition that $(2af)^2 = (b^2 - 4ac)(g^2)$. So, it is enough to show that the existence of $f, g \in A$ with $g \neq 0$ and $(2af)^2 = (b^2 - 4ac)(g^2)$ is equivalent to $b^2 - 4ac$ being a square in A. The $\Leftarrow$ direction here is evident, so, it is enough to prove the $\Rightarrow$ direction:

So, assume that there are $f, g \in A$ with $g \neq 0$ and $(2af)^2 = (b^2 - 4ac)(g^2)$. Since A is a UFD, we see that the irreducible factors of g are dividing $2af$, and therefore $2af/g \in A$. So, with $s = 2af/g$ we have $s^2 = b^2 - 4ac$.

2. In all the points we take $A = \mathbb{C}[y]$, as then $A[x] = \mathbb{C}[x, y]$.

(a) This is irreducible, because

$$b^2 - 4ac = 4y^2 - 4 = 4(y^2 - 1) = 4(y + 1)(y - 1)$$

is not a square (no double irreducible factors).

(b) This polynomial is not primitive (y divides all the coefficients, so actually point 1.) does not apply to it. In fact, the polynomial is not irreducible as it is divisible by y.

After dividing by y we have: $(y + 1)x^2 + x + y$, so we have $b^2 - 4ac = 1 - 4y(y + 1) = -4y^2 - 4y + 1$. The discriminant of this polynomial is $(-4)^2 - 4(-4) = 32$, which is not zero, and hence $-4y^2 - 4y + 1$ does not have a double root, and hence it is not a square in $\mathbb{C}[y]$. So, we get that $(y + 1)x^2 + x + y$ is irreducible. Hence,

$$y((y + 1)x^2 + x + y) = y^2x^2 + yx^2 + yx + y^2$$

is the irreducible decomposition.

(c) As in point 2.a.) we have $b^2 - 4ac = y^2 - 4y^2 = -3y^2$, which is the square of $\sqrt{-3} y in \mathbb{C}[y]$. So the polynomial is not irreducible. From the above proof we see that the fatorization is

$$(x + y(1 + i\sqrt{3})/2)(x + y(1 - i\sqrt{3})/2)$$

, or with other words

$$(x + ey)(x + e^2y)$$

where e is a 3rd root of unity.

Remark: This looks non-symmetric in x and y, but it really is, as it funnily equals $(e^2x + y)(ex + y)$, and as predicted by the UFD property, the factors of this second decomposition are associated to the factors of the first one.