Solutions to Homework 7: 26 April 2022
CS-526 Learning Theory

Problem 1

1) For every $i \in [K]$, $\underline{d}_i$ is the i^{th} canonical basis vector of $\mathbb{R}^K$ and we define the latent random vector $\underline{h} \in \{\underline{d}_i : i \in [K]\}$ whose distribution is $\forall i \in [K] : \mathbb{P}(\underline{h} = \underline{d}_i) = w_i$. Finally, let $\underline{x} = \sum_{i=1}^K h_i \underline{a}_i + \underline{z}$ where $\underline{z} \sim \mathcal{N}(0, \sigma^2 I_{D\times D})$ is independent of $\underline{h}$. The random vector $\underline{x}$ has a probability density function $p(\cdot)$. We have:

$$\mathbb{E}[\underline{x}] = \sum_{i=1}^K \mathbb{E}[h_i]\underline{a}_i + \mathbb{E}[\underline{z}] = \sum_{i=1}^K w_i\, \underline{a}_i \quad ;$$

$$\begin{aligned}\mathbb{E}[\underline{x}\underline{x}^T] &= \mathbb{E}[\underline{z}\underline{z}^T] + \sum_{i=1}^K \mathbb{E}[h_i] \underbrace{\mathbb{E}[\underline{z}]}_{=0} \underline{a}_i^T + \mathbb{E}[h_i]\underline{a}_i\mathbb{E}[\underline{z}]^T + \sum_{i,j=1}^K \underbrace{\mathbb{E}[h_i h_j]}_{=w_i\delta_{ij}} \underline{a}_i\underline{a}_j^T \\ &= \sigma^2 I_{D\times D} + \sum_{i=1}^K w_i\, \underline{a}_i\underline{a}_i^T\ .\end{aligned}$$

Finally, to compute the third moment tensor, note that $\mathbb{E}[\underline{z} \otimes \underline{z} \otimes \underline{z}] = 0$ and that for every $(i,j) \in [K]^2$: $\mathbb{E}[\underline{a}_i \otimes \underline{a}_j \otimes \underline{z}] = \mathbb{E}[\underline{a}_i \otimes \underline{z} \otimes \underline{a}_j] = \mathbb{E}[\underline{z} \otimes \underline{a}_i \otimes \underline{a}_j] = 0$. Hence:

$$\begin{aligned}\mathbb{E}[\underline{x} \otimes \underline{x} \otimes \underline{x}] = &\sum_{i,j,k=1}^K \underbrace{\mathbb{E}[h_i h_j h_k]}_{=w_i\delta_{ij}\delta_{ik}} \underline{a}_i \otimes \underline{a}_j \otimes \underline{a}_k \\ &+ \sum_{i=1}^K \mathbb{E}[h_i]\mathbb{E}[\underline{a}_i \otimes \underline{z} \otimes \underline{z}] + \mathbb{E}[h_i]\mathbb{E}[\underline{z} \otimes \underline{a}_i \otimes \underline{z}] + \mathbb{E}[h_i]\mathbb{E}[\underline{z} \otimes \underline{z} \otimes \underline{a}_i] \\ = &\sum_{i=1}^K w_i\, \underline{a}_i \otimes \underline{a}_i \otimes \underline{a}_i + \sigma^2 \sum_{j=1}^D \sum_{i=1}^K w_i (\underline{a}_i \otimes \underline{e}_j \otimes \underline{e}_j + \underline{e}_j \otimes \underline{e}_j \otimes \underline{a}_i + \underline{e}_j \otimes \underline{a}_i \otimes \underline{e}_j)\,.\end{aligned}$$

2) Let $A = [\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_K] \in \mathbb{R}^{D\times K}$ and $A' = [\underline{a}'_1, \underline{a}'_2, \ldots, \underline{a}'_K] \in \mathbb{R}^{D\times K}$. By definition, $\widetilde{R} = \Sigma^{-1}R\Sigma$ where Σ is the diagonal matrix such that $\Sigma_{ii} = \sqrt{w_i}$ and $A' = A\widetilde{R}^T$. We can directly apply the formula of question 1) to compute the second moment matrix of the new mixture of Gaussians:

$$\begin{aligned}\mathbb{E}[\underline{x}\underline{x}^T] &= \sigma^2 I_{D\times D} + A'\Sigma^2 A'^T = \sigma^2 I_{D\times D} + A\widetilde{R}^T\Sigma^2\widetilde{R}A^T \\ &= \sigma^2 I_{D\times D} + A\Sigma R^T R\Sigma A^T = \sigma^2 I_{D\times D} + A\Sigma^2 A^T\ .\end{aligned}$$

Problem 2: Examples of tensors and their rank

1) The matrices corresponding to B, P, E are:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ ;\ P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\ ;\ E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The frontal slices of G and W are:

$$G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; G_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \; ; \; W_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} .$$

2) B and E are clearly rank-2 matrices, while $P = (e_0 + e_1) \otimes (e_0 + e_1)$ is a rank-1 matrix.
By its definition, G is at most rank 2. Assume it is rank 1: $G = a \otimes b \otimes c$ with $a, b, c \in \mathbb{R}^2$. We have $a_1 b_1 c_1 = G_{111} = 1$ and $a_2 b_1 c_1 = G_{211} = 0$ so we must have $a_2 = 0$. Besides, $a_2 b_2 c_2 = G_{222} = 1$ and $a_1 b_2 c_2 = G_{122} = 0$ so $a_1 = 0$. Hence $a^T = (0, 0)$ and G is the all-zero tensor. This is a contradiction and we conclude that G is rank 2.
By its definition, W is at most rank 3. To prove the rank cannot be smaller than 3, we will proceed by contradiction:

- Assume W is rank 1: $W = a \otimes b \otimes c$ with $a, b, c \in \mathbb{R}^2$. We have $a_1 b_1 c_1 = W_{111} = 0$ and $a_2 b_1 c_1 = W_{211} = 1$ so $a_1 = 0$. Besides, $a_1 b_1 c_2 = W_{112} = 1$ and $a_2 b_1 c_2 = W_{212} = 0$ so $a_2 = 0$. Then $a = (0, 0)^T$ and W is the all-zero tensor, which is a contradiction.

- Assume W is rank 2: $W = a \otimes b \otimes c + d \otimes e \otimes f$. We claim that *$a$ and d must be linearly independent*. Indeed, suppose they are parallel and take a vector x perpendicular to both a and d. Then
$$W(x, I, I) = (x^T a) b \otimes c + (x^T d) e \otimes f = 0$$
but also
$$W(x, I, I) = (x^T e_0) e_0 \otimes e_1 + (x^T e_0) e_1 \otimes e_0 + (x^T e_1) e_0 \otimes e_0 = \begin{bmatrix} x^T e_1 & x^T e_0 \\ x^T e_0 & 0 \end{bmatrix}$$
which cannot be zero since x cannot be perpendicular to both e_0 and e_1. Now, we take x perpendicular to d. We have
$$W(x, I, I) = (x^T a) b \otimes c$$
which is rank one. Therefore, we must have $x^T e_0 = 0$ which implies that x is parallel to e_1 and thus $\underline{d \text{ parallel to } e_0}$. Now, if we take x perpendicular to a, the matrix
$$W(x, I, I) = (x^T d) e \otimes f$$
is rank one and, once again, we must have $x^T e_0 = 0$, which implies x parallel to e_1 and thus $\underline{a \text{ parallel to } e_0}$. Hence, we have shown that a and d are linearly independent but also that both are parallel to e_0. This is a contradiction.

3) We expand the tensor products in the definition of D_ϵ:

$$\begin{aligned} D_\epsilon &= \frac{1}{\epsilon} \Big[(e_0 + \epsilon e_1) \otimes (e_0 + \epsilon e_1) \otimes (e_0 + \epsilon e_1) - e_0 \otimes e_0 \otimes e_0 \Big] \\ &= \frac{1}{\epsilon} \Big[e_0 \otimes e_0 \otimes e_0 + \epsilon\, e_0 \otimes e_0 \otimes e_1 + \epsilon\, e_0 \otimes e_1 \otimes e_0 + \epsilon\, e_1 \otimes e_0 \otimes e_0 \\ &\qquad + \epsilon^2\, e_1 \otimes e_1 \otimes e_0 + \epsilon^2\, e_1 \otimes e_0 \otimes e_1 + \epsilon^2\, e_0 \otimes e_1 \otimes e_1 + \epsilon^3\, e_1 \otimes e_1 \otimes e_1 - e_0 \otimes e_0 \otimes e_0 \Big] \\ &= e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0 \\ &\qquad + \epsilon (e_1 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_1) + \epsilon^2\, e_1 \otimes e_1 \otimes e_1 \\ &= W + \epsilon (e_1 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_1) + \epsilon^2\, e_1 \otimes e_1 \otimes e_1 \, . \end{aligned}$$

Hence $\lim_{\epsilon \to 0} D_\epsilon = W$.

Problem 3

1) There cannot be an analogous general result for tensors. Indeed, the order-3 tensor W of Problem 2 is rank 3 and we showed in 3) that $\lim_{\epsilon\to 0}\|W - D_\epsilon\|_F = 0$. So there is no minimum attained in the space of rank 2 tensors. In this sense, there is simply no *best* rank-two approximation of W.

2) Let M a matrix of rank $R+1$ with singular values $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_R \geq \sigma_{R+1} > 0$. By the Eckart-Young-Mirsky theorem, the minimum of $\|M - \widehat{M}\|_F$ over all the matrices $\widehat{M}$ of rank less than, or equal to, R is $\sigma_{R+1} > 0$. Therefore, there cannot be a sequence of matrices M_n given by a sum of R rank-one matrices such that $\lim_{n\to+\infty}\|M - M_n\|_F = 0$.

Now let $M \in \mathbb{C}^{M\times N}$ be a matrix of rank $R-1$ with $R \leq \min\{M, N\}$. Let $M = U\Sigma V^*$ be the SVD of M where $\sigma_1 \geq \cdots \geq \sigma_{R-1} > 0$ are its singular values. For all positive integer n, we define $\sigma_R^{(n)} := \sigma_{R-1}/n$ as well as the rank-R matrix $M_n = U\Sigma_n V^*$ where Σ_n is a $M \times N$ diagonal matrix whose nonzero diagonal entries are $\sigma_1 \geq \cdots \geq \sigma_{R-1} \geq \sigma_R^{(n)}$. Clearly $\lim_{n\to+\infty}\|M - M_n\|_F = \lim_{n\to+\infty}\frac{\sigma_{R-1}}{n} = 0$. A similar procedure can be applied if M is a tensor.

3) In the real-valued case, we have:

$$|T(R_1, R_2, R_3)^{\alpha\beta\gamma}|^2 = \sum_{\delta,\epsilon,\zeta,\delta',\epsilon',\delta'} R_1^{\delta\alpha} R_1^{\delta'\alpha} R_2^{\epsilon\beta} R_2^{\epsilon'\beta} R_3^{\zeta\gamma} R_3^{\zeta'\gamma} T^{\delta\epsilon\zeta} T^{\delta'\epsilon'\zeta'} \ .$$

Summing over α, β, γ and using the orthogonality of rotation matrices, we find:

$$\sum_\alpha R_1^{\delta\alpha} R_1^{\delta'\alpha} = \delta_{\delta\delta'}, \quad \sum_\beta R_2^{\epsilon\beta} R_2^{\epsilon'\beta} = \delta_{\epsilon\epsilon'}, \quad \sum_\gamma R_3^{\zeta\gamma} R_3^{\zeta'\gamma} = \delta_{\zeta\zeta'} \ .$$

The result directly follows:

$$\begin{aligned}\|T(R_1, R_2, R_3)\|_F^2 &= \sum_{\alpha,\beta,\gamma} |T(R_1, R_2, R_3)^{\alpha\beta\gamma}|^2 \\ &= \sum_{\delta,\epsilon,\zeta,\delta',\epsilon',\delta'} \delta_{\delta\delta'}\delta_{\epsilon\epsilon'}\delta_{\zeta\zeta'} T^{\delta\epsilon\zeta} T^{\delta'\epsilon'\zeta'} \\ &= \sum_{\delta\epsilon\zeta} |T^{\delta\epsilon\zeta}|^2 \\ &= \|T\|_F^2 \ .\end{aligned}$$