Exercice 7. 1. As $\deg f = 3$ one just has to verify that f does not have a root over $\mathbb{Q}$. So, we need to show that if a and b are non-zero relatively prime integers, then

$$(a/b)^3 + (a/b) + 1 \neq 0$$

, or equivalently

$$a^3 + ab^2 + b^3 \neq 0.$$

Suppose the contrary. Then b divides a^3 and a divides b^3. Using the relative prime assumption we obtain both a and b are plus-minus 1, but one cannot add together three numbers, each plus or minus 1 to get 0.

2. Let α, β and γ be the three roots of f in its splitting field. Assume that they are all real. Then we have

$$f = (x-\alpha)(x-\beta)(x-\gamma)$$

and hence

$$\alpha + \beta + \gamma = 0$$

and

$$\alpha\beta + \alpha\gamma + \beta\gamma = a$$

From the first equation we have $\gamma = -\alpha - \beta$. Plugging this into the left side of the second equation yields

$$\alpha\beta + \alpha(-\alpha-\beta) + \beta(-\alpha-\beta) = -\alpha^2 - \beta^2 - \alpha\beta = -\frac{1}{2}(\alpha+\beta)^2 - \frac{\alpha^2}{2} - \frac{\beta^2}{2} \leq 0$$

However, we assumed that $a > 0$. This is a contradiction.

3. As $\deg f = 3$, and complex roots of a real polynomial come in complex conjugate pairs, f has to have a real root. Let this real root be α. Then, $\mathbb{Q} \subseteq \mathbb{Q}(\alpha)$ is a degree 3 extension and additionally $\mathbb{Q}(\alpha) \subseteq \mathbb{R}$. Hence, the other two roots of f, say β and γ, cannot be contained in $\mathbb{Q}(\alpha)$. So, every element $g \in \mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ can send α only to α. However, as α generated $\mathbb{Q}(\alpha)$ this means that $g = \mathrm{id}$.

4. Let α, β and γ be as in the previous point. Then both β and γ are roots of $h = \frac{f}{x-\alpha} \in \mathbb{Q}(\alpha)[x]$. As this polynomial has degree 2, and β and γ are not in $\mathbb{Q}[x]$, $h = m_{\beta,\mathbb{Q}(\alpha)} = m_{\gamma,\mathbb{Q}(\alpha)}$. So, $\mathbb{Q}(\alpha,\beta,\gamma)$ has degree 2 over $\mathbb{Q}(\alpha)$. So, by the multiplicativity of the degrees of field extensions, $L = \mathbb{Q}(\alpha,\beta,\gamma)$ has degree 6 over $\mathbb{Q}$. Let G be the Galois group of L over $\mathbb{Q}$. Then, G acts faithfully on α, β and γ, which yields an embedding $G \hookrightarrow S_3$. As both have 6 elements, this is in fact an isomorphism.