Introduction to Differentiable Manifolds

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Exercise Series 1 - Topological and smooth manifolds **2022–09–20**

Convention: We understand a subset/product/quotient of topological space(s) to be automatically endowed with the subspace/product/quotient topology unless we state otherwise.

Exercise 1.1. Which of the following spaces are locally Euclidean? Which are (globally) homeomorphic to some Euclidean space?

(a) an open ball in $\mathbb{R}^n$
(b) the closed interval $[0, 1] \subset \mathbb{R}$
(c) the circle $\mathbb{S}^1 \subset \mathbb{R}^2$
(d) the zero set of the function $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = xy$
(e) the "bent line" $\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, xy = 0\}$.

Exercise 1.2. If a space M is locally Euclidean of dimension n at some point p, show that p has an open neighborhood that is homeomorphic to the whole space $\mathbb{R}^n$, or to a open ball $B_r(x)$.

Deduce the equivalent definitions of topological n-manifold.

Exercise 1.3. The **line with two origins** is the space M obtained as quotient of the space $X = \{\pm 1\} \times \mathbb{R}$ by the equivalence relation $(i, x) \sim (j, y)$ iff $x = y \neq 0$.

(a) Show that M is locally Euclidean and second countable, but not Hausdorff.
(b) Find a sequence of points in M that converges to two different points, and show that this cannot happen in a Hausdorff space.

Exercise 1.4. Let N be an open subset of a topological n-manifold M.

(a) Show that N is a topological n-manifold.
(b) Show that any smooth structure $\mathcal{A}$ on M determines a smooth structure $\mathcal{B}$ on N, consisting of the charts $(U, \varphi) \in \mathcal{A}$ such that $U \subseteq N$.

Exercise 1.5. Show that the product of two topological manifolds is a topological manifold. What is its dimension?

Exercise 1.6. We have seen in the lecture that $\mathbb{S}^n$ is a topological n-manifold. Show that the charts $(U_i^{+,-}, \varphi^{+,-})_{i=1,\dots,n}$ form a smooth atlas for $\mathbb{S}^n$.

Exercise 1.7 (To hand in). Show that the **projective space** $\mathbb{P}^n$, defined as the quotient of $\mathbb{R}^{n+1}\backslash\{0\}$ by the equivalence relation $x \sim y$ iff $x = \lambda\, y$ for some $\lambda \in \mathbb{R}\backslash\{0\}$, is a smooth n-manifold with atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i=0,\dots,n}$ given by

$$U_i := \{[x] \in \mathbb{P}^n \mid x_i \neq 0\}, \qquad \varphi_i([x]) = \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right),$$

where $[x] \in \mathbb{P}^n$ denotes the equivalence class of a point $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}\backslash\{0\}$.

Exercise 1.8. Show that the **n-torus** $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, defined as the quotient of $\mathbb{R}^n$ by the equivalence relation $x \sim y$ iff $y - x \in \mathbb{Z}^n$, is a topological n-manifold.

Exercise 1.9. Show that $(\mathbb{R}, \mathrm{id}_{\mathbb{R}})$ and $(\mathbb{R}, \psi : x \mapsto x^3)$ define to different smooth structures on the real line.