Lecture 2

§2.0 Last time

- A smooth manifold is a pair $(M, \mathcal{A})$ where
- M is a topological n-mfd
- $\mathcal{A}$ is a smooth atlas (i.e. $\overbrace{\{(U_i, \varphi_i : U_i \to \hat{U}_i \subseteq \mathbb{R}^n\}}^{\text{choices}}$ coord. charts s.t. - $\bigcup U_i = M$
  - $\varphi_j \circ \varphi_i^{-1}$ transition function are diffeomorphism)

- A smooth atlas $\mathcal{A} \subseteq \bar{\mathcal{A}}$ maximal uniquely $\Rightarrow$ Fixing $\mathcal{A}$ determines a smooth structure

Today

§2.1 Smooth functions and atlas

§2.2 Partition of unity and Bump Function

§2.1

Remark: Everything I say can be generalised in two ways: 1. we can work with C^k-manifolds;
2. We can talk about $C^{k'}$-smooth functions / maps among C^k mfds $\forall$ $k' \leq k$.

Def: ① A function $F: M \to \mathbb{R}^k$ is smooth at $p \in M$ if $\exists$ a local chart $(U, \varphi) \in \mathcal{A}$ s.t.

$$\boxed{F|_U \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}^k} \text{ is}$$

smooth at $\varphi(p)$

coordinate expression or
local expression for F

② given $(M, \mathcal{A})$, $(N, \mathcal{B})$ smooth manifolds a map $F: M \to N$ is smooth at p

if $\exists$ Ⓐ a local chart $p \in (U, \varphi)$
Ⓑ " " " $F(p) \in (V, \psi)$ and $F(U) \subseteq V$

s.t.

$$\boxed{\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)}$$

is smooth at $\varphi(p)$

We say F smooth if it is smooth at every p

③ F is a diffeomorphism is it is smooth, bijective and F^{-1} is smooth

Proposition

If $F: M \to N$ is smooth at p

$\Rightarrow$ ~~the~~ Every local expression $F|_{\varphi}^{\psi} = \psi \circ F \circ \varphi^{-1}$ is a smooth map from a pen in $\mathbb{R}^{n}$ to a a open of $\mathbb{R}^{m}$

proof

Consider (U, φ), (V, ψ) two pair of local
$(\tilde{U}, \tilde{\varphi})$ $(\tilde{V}, \tilde{\psi})$ chart satisfying

the condition of the definition

$\Rightarrow$ $\cdot$ $U \cap \tilde{U}$ is a nbd of p and

$\cdot$ $V \cap \tilde{V}$ " " of $F(p)$ s.t. $F(U \cap \tilde{U}) \subseteq V \cap \tilde{V}$

and the two local expressions are related as follows

$$\boxed{\tilde{\psi} \circ F \circ \tilde{\varphi}^{-1}\Big|_{\tilde{\varphi}(U \cap \tilde{U})} = \underbrace{(\tilde{\psi} \circ \psi^{-1})}_{\substack{\text{smooth} \\ \psi(V \cap \tilde{V}) \subseteq \mathbb{R}^{K} \\ \tilde{\psi}(V \cap \tilde{V}) \subseteq \mathbb{R}^{K}}} \underbrace{(\psi \circ F \circ \varphi^{-1}\Big|_{\varphi(U \cap \tilde{U})})}_{\substack{\text{smooth} \\ \varphi(U \cap \tilde{U}) \to \psi(V \cap \tilde{V})}} (\varphi \circ \tilde{\varphi}^{-1})}$$

change of local expression formula

smooth function $\tilde{\varphi}(U \cap \tilde{U}) \subseteq \mathbb{R}^{n}$ to $\varphi(U \cap \tilde{U}) \subseteq \mathbb{R}^{n}$

∎

Examples

① $id : (M, A) \to (M, A)$ is smooth :

Take any $p \in M$, (U, φ) a smooth chart

$\Rightarrow id_\varphi^\varphi : \varphi(U) \to \varphi(U)$

$\varphi \circ id \circ \varphi^{-1}$ ☒

Rm : The id of M as a topological space is different than the id as a map of smooth manifolds

If A_1, A_2 are two atlases on M and $I : (M, A_1) \to (M, A_2)$ is the id on underlying top spaces

$\Rightarrow I$ is smooth $\Leftrightarrow$ the local expressions $I_\varphi^\psi = \psi \circ \varphi^{-1}$ are smooth

$\Leftrightarrow A_1, A_2$ are smoothly compatible.

② $c : M \to N$ constant map

③ $\iota : U \hookrightarrow M$ inclusion of a open submanifold

④ composition $G \circ F$ of smooth maps

Prove these claims

5) $F : \mathbb{B}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{B}^n$

$x \longmapsto \frac{x}{\sqrt{(1-\|x\|^2)}}$ $y \longrightarrow \frac{1}{\sqrt{(1+\|y\|^2)}}$

give a diffeomorphism of the open ball and $\mathbb{R}^n$

(6) $F : (\mathbb{R}, id) \to (\mathbb{R}, \psi : x \to x^3)$

where F is defined on the underlying topological spaces as $x \to x^{1/3}$

$\Rightarrow$ F is a diffeomorphism

Proposition : ~~a~~ map $F : M \to N$ is smooth

$\Leftrightarrow$ For any open subset $U \subseteq M$ $F|_U : U \to F(U)$ is smooth

i.e smoothness is a local property!

proof

$\Rightarrow$ Follows from ③, ④

$\Leftarrow$ Just use the definition of smoothness choosing U to be the coordinate opens ∎

Theorem

Let M, N diffeomorphic manifolds

$\Rightarrow \dim M = \dim N$

proof

F diffeomorphism $\Rightarrow$ F, F^{-1} smooth

$\Rightarrow \forall$ (U, φ), (V, ψ) local charts with $F(U) \subseteq V$

$G = \psi \circ F \circ \varphi^{-1}$ are smooth from

$G^{-1} = \varphi \circ F^{-1} \circ \psi^{-1}$

$\varphi(U) \subseteq \mathbb{R}^{\dim M} \longrightarrow \psi(V) \subseteq \mathbb{R}^{\dim N}$

$\Rightarrow$ Using the chain rule

$DG(a)$, $DG^{-1}(G(a))$ are linear maps which are inverse to each other

since $D(G^{-1} \circ G) = D(Id_{\mathbb{R}^{\dim M}})$ $\Rightarrow n = m$

$D(G \circ G^{-1}) = D(Id_{\mathbb{R}^{\dim N}})$ $\blacksquare$

Remark

We will denote by $C^\infty(M)$ the ring of smooth functions $f: M \to \mathbb{R}$.

In general, you might be interested if smooth functions which are only defined on some open $U \subseteq M$.

$\mathcal{C}^\infty(M) : \{ U \subseteq M \text{ open} \} \to \text{sets}$

$U \longmapsto C^\infty(U)$

sheaf

§ 2.2 Partition of unity and smooth bump functions

partition of unity is an important tool to turn local constructions / definition on manifolds into global ones

Recall

given a continuous function $\eta : M \to \mathbb{R}$ the support of η is $\operatorname{supp}(\eta) = \overline{\{p \in M \mid \eta(p) \neq 0\}}$ [closed subset]

[**Definition**: Let $\mathcal{U} = \{U_i\}_{i \in I}$ a open cover of M. A (smooth) partition of unity (POU) subordinate to $\mathcal{U}$ is a collection of smooth functions $\eta_i : M \to \mathbb{R}$ s.t.

(i) $0 \leq \eta_i(x) \leq 1 \quad \forall\ i \in I,\ x \in X$

(ii) $\operatorname{supp} \eta_i \subseteq U_i$ (subordinate to $\mathcal{U}$)

(iii) $\forall\ x \in M$, $\exists$ a nbd U s.t. all but finitely many η_i vanish on U

[$\Leftrightarrow$ the collection of subsets $\{\operatorname{supp}(\eta_i)\}$ is locally finite]

(iv) $\sum_{i \in I} \eta_i(x) = 1 \quad \forall\ x \in M$

Theorem

Let M be a smooth manifold. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ any indexed open cover

$\Rightarrow \exists$ a smooth partition of unity subordinate to $\mathcal{U}$.

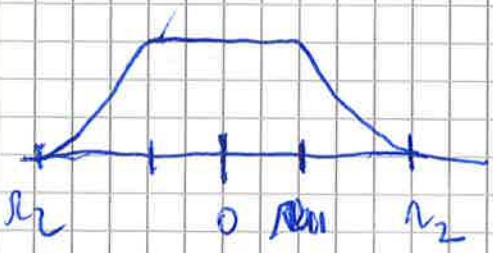
• **Preliminaries from analysis:** smooth bump functions.

Proposition: Given $0 < r_1 < r_2$ there exist a smooth function

$$H : \mathbb{R}^n \to \mathbb{R}$$

s.t.

$$H = \begin{cases} 1 & \text{on } \overline{B_{r_1}(0)} \qquad \|x\| \le r_1 \\ 0 < H(x) < 1 & B_{r_2}(0) \setminus \overline{B_{r_1}(0)} \\ 0 & \mathbb{R}^n \setminus \overline{B_{r_2}(0)} \qquad \|x\| > r_2 \end{cases}$$



Proof: Define

$$H(x) = g(\|x\|) \quad \text{where}$$

$g : \mathbb{R} \to [0,1]$ is the cut off function

$$g(t) = \begin{cases} \dfrac{f(r_2 - t)}{f(r_2 - t) + f(t - r_2)} & r_1 < t < r_2 \\ 0 & t > r_2 \\ 1 & t < r_1 \end{cases}$$

and $F(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$

Then it is all a consequence of smoothness of F $\blacksquare$

Preliminaries from topology

- A collection $\mathcal{X}$ of subsets of M is locally finite if $\forall p \in M$ $\exists$ a nbd $p \in U$ s.t. $U \cap X_i \neq \emptyset$ for finitely many i.
- given a open cover $\mathcal{U}$ of M a refinement $\mathcal{V} \subseteq \mathcal{U}$ is a open cover s.t. $\forall V \in \mathcal{V}$ $\exists$ $U \in \mathcal{U}$ with $V \subseteq U$
- M is said paracompact if any open cover admits a locally finite refinement

$\boxed{\text{Theorem : PARACOMPACTNESS}}$ second countability

Every topological manifold is paracompact.

In fact, given a open cover $\mathcal{U}$ and $\mathcal{B}$ a base for the topology

$\Rightarrow \exists$ a countable, locally finite open refinement with elements in $\mathcal{B}$

Remark
We can take $\mathcal{B}$ to be a family of coordinate balls i.e $\varphi^{-1}(B_r(\varphi(x)))$ - such that their closure in M is compact $\varphi^{-1}(\overline{B_r}(\varphi(x)))$

Proof of Thm :

- Each U_α is a smooth manifold and it has a basis $\mathcal{B}_\alpha$ of coordinate balls for its topology.
  $\mathcal{B}_\alpha$ is countable since U_α is 2nd count.
- $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha$ is a basis for the top. of M

$\Rightarrow$ • The open cover $\mathcal{U}$ admits a locally finite, countable refinement $\{B_i\}$ where $B_i \in \mathcal{B}$.

• We can choose the B_i in such a way that $\overline{B_i}$ is compact in M.

- Notice furthermore that if $\{B_i\}$ is locally finite
$\Rightarrow$ also $\{\overline{B_i}\}$ is locally finite.

- By definition of refinement B_i is a coord ball in some open U_α
$\Rightarrow$ we can always find $B_i' \subseteq U_\alpha$ s.t.
$\overline{B_i} \subsetneq B_i'$ $\Leftrightarrow$ if we denote by

$\varphi_i : U_i \subseteq U_\alpha \to \mathbb{R}^n$ the local coord. chart

$$\Rightarrow \quad \varphi_i(B_i) = B_{r_i}(0)$$
$$\varphi_i(B_i') = B_{r_i'}(0) \quad \text{with} \quad \underline{r_i' > r_i > 0}$$

- Define $f_i : M \to \mathbb{R}$ to be

$$f_i = \begin{cases} H \circ \varphi_i & \text{on } B_i' \\ 0 & \text{on } M \setminus \overline{B_i} \end{cases}$$

- $\operatorname{supp}(f_i) = \overline{B_i}$

- Define $f(x) = \sum_i f_i(x)$ well defined by local finiteness of the $\overline{B_i}$

- $f(x) \geq 0$
$\Rightarrow$ Define $\boxed{g_i(x) = f_i(x)/f(x)}$

$0 \leq g_i \leq 1$

$\sum_i g_i = 1$

- subordination condition

$\{B_i\}$, $\{B_i'\}$ are refinements of U_α

$\Rightarrow$ we have a function

$$e : I \to A$$
$$\alpha(i) \to e(i) \text{ if } B_i' \subseteq V_{e(i)}$$

$$\boxed{\psi(\alpha) = \sum_{i : e(i) = \alpha} g_i}$$ is a

smooth partition of unity. ∎

<u>Corollary</u>

Let $F : A \to \mathbb{R}$ a smooth functions with $A \subseteq M$ closed subset and $A \subseteq U \subseteq M$ open

$\Rightarrow \exists \tilde{F} : M \to \mathbb{R} \mid \tilde{F}$ is smooth $\tilde{F}|_A = F$

$\text{supp}(\tilde{F}) \subseteq U$

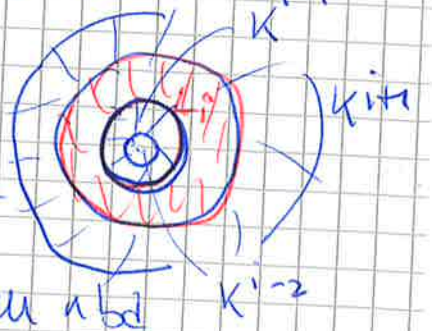
F can be extended to a smooth function on the open proof of paracompactness theorem

- If M is n-topological manifold
$\Rightarrow$ it has a basis of open subset B_i
s.t. $\overline{B_i}$ is compact in M (take coordinate balls)

- since M is second countable $\Rightarrow$ we can refine the cover $\{B_i\}_{i\in I}$ to a countable one $\{B_i\}_{i\in\mathbb{N}}$

- $\Rightarrow \exists\ (K_i)_{i=1}^{\infty}$ sequence of compact subsets
s.t. $X=\bigcup K_i$, $K_i \subseteq \mathrm{Int}(K_{i+1})$
  - take $K_1=\overline{B_1}$
  - $K_j=\overline{U_1\cup\dots\cup U_{m_j}}$ where (K_{j-1} is covered by U_{m_i} (finitely many!)) ☒

- take $\mathcal{U}$ cover, $\mathcal{B}$ the coord. balls base
- take (K_i) a comp. ex.



$\Rightarrow$
  - $L_i=K_i\setminus \mathrm{Int}(K_{i-1})$ is compact
  - $W_i=\mathrm{Int}\,K_{i+1}\setminus K_{i-2}$ is a open nbd of L_i

- $\forall x\in L_i$, let $U_x\subseteq \mathcal{U}$ a open cont. x
$\Rightarrow$ we have a B_x^i coor. ball s.t. $\overline{B_x^i}\subseteq U\cap W_i$

- since L_i is compact $\exists$ finite family $(B^i_{x_j})_J$ covering L_i
- Repeating $\forall i$ $(B^i_{x_j})_{i,J}$ is countable and covers M, and $\overline{B^i_{x_j}} \subseteq U_{x_j} \cap W_i$

$\Longrightarrow \forall x$, $x \cap W_\ell$ only for $|\ell - i| \leq 1$

$$\begin{cases} \ell = i+1 \\ \ell = i \\ \ell = i-1 \end{cases}$$

$\blacksquare$