Introduction to Differentiable Manifolds	
EPFL – Fall 2022	F. Carocci, M. Cossarini
Exercise Series 5 - Submanifolds	**2022–10–18**

Exercise 5.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^3 + y^3 + 1$.

(a) What are the regular values of f? For which $c \in \mathbb{R}$ is the level set $f^{-1}(\{c\})$ an embedded submanifold of $\mathbb{R}^2$?

(b) In the case where $S = f^{-1}(\{c\})$ is an embedded submanifold, $p \in S$, write down an equation for the tangent space $\iota_*(\mathrm{T}_p S) \subset \mathrm{T}_p\mathbb{R}^2$ where as usual we identify $T_p\mathbb{R}^2 \cong \mathbb{R}^2$ (i.e. you are expected to write down the equation for a line in $\mathbb{R}^2$).

Exercise 5.2. Let $S = F^{-1}(c)$ for c a regular value of a smooth function $F : M \to N$. Let us fix $p \in S$. Prove that $T_pS = \mathrm{Ker}(D_pF : T_pM \to T_F(p)N)$.

Hint: Use the Slice chart Lemma and the fact that $T_pM \cong T_pU \cong T_{\varphi(p)}\varphi(U)$ for every open neighbourhood U of p and for any smooth chart φ

Exercise 5.3. Show that the map $g : \mathbb{T}^2 \to \mathbb{R}^3$ given by

$$g([s, t]) = ((2 + \cos s) \cos t, (2 + \cos s) \sin t, \sin s)$$

is a smooth embedding of the 2-torus in $\mathbb{R}^3$.
(In this case the torus is defined as $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$.)

Exercise 5.4 (To hand in). Show that the following subgroups of $GL_n(\mathbb{R})$ are closed submanifolds. Compute their dimension and their tangent space at the identity.

(a) The *special linear group* $\mathrm{SL}_n(\mathbb{R})$, consisting of matrices with determinant equal to 1.

(b) The *orthogonal group* $O_n(\mathbb{R})$, consiting of the orthogonal matrices A (which satisfy $A^\top A = I_n$).

Hint: Consider the map $f : M_n \to M_n^{sym}$ that sends $A \mapsto A^\top A$, there M_n^{sym} is the vector space of *symmetric* $n \times n$ matrices.

Exercise 5.5. If S_0, S_1 are smooth embedded submanifolds of M_0, M_1 respectively, then $S_0 \times S_1$ is a smooth embedded submanifold of $M_0 \times M_1$.

Exercise 5.6. (a) Show that a subset $S \subseteq \mathbb{R}^n$ is a smooth-embedded k-submanifold if each point $x \in S$ has an open neighborhood W such that the set $S \cap W$ is the graph of a smooth function that expresses some $n - k$ coordinates in terms of the remaining k coordinates. (More precisely, the function is of the form $f : U \subseteq \mathbb{R}^I \to \mathbb{R}^{I'}$, where I is a k-element subset of $n := \{0, \dots, n-1\}$, I' is its complement, and $U \subseteq \mathbb{R}^I$ is an open set.)

(b) Let S be the set of real $m \times n$ matrices of rank k. Show that S is a smooth submanifold of $\mathbb{R}^{m\times n}$. What is its dimension ?

Hint: A rank-k matrix $A \in \mathbb{R}^{m\times n}$ has an invertible $k \times k$ submatrix $A|_{I\times J}$ (where $I \subseteq m$, $J \subseteq n$ are k-element sets). Show that the coefficients $A_{i',j'}$ with $i' \notin I$ and $j' \notin J$ can be expressed as a smooth function of the other coefficients of A.

Exercise 5.7. If M is connected and $f : M \to M$ is an idempotent smooth map ("idempotent" means that $f \circ f = f$), then $f(M)$ is an embedded submanifold of M.
Hint: Show that f has constant rank. Use what you know about a linear projector $P : V \to V$ and the complementary projector $\mathrm{id}_V - P$.