Exercise 7.1. Show that the Moëbious bundle as defined in the lecture is a smooth vector bundle on S^1 (Exhibit local trivialization and compute the transition functions.)

Exercise 7.2. Show that the tangent bundle TS^1 is trivial.

Exercise 7.3 (Properties of smooth vector fields)**.** Let M be a smooth manifold and let $X : M \to TM$ be a vector field. Show that the following are equivalent:

(a) X is a smooth vector field.
(b) The component functions of X are smooth with respect to all charts of one particular smooth atlas of M.
(c) For any smooth function $f : U \to \mathbb{R}$ on an open set $U \subset M$, the function $Xf : U \to \mathbb{R}$ defined by $Xf(p) := X_p(f)$ is smooth.

Exercise 7.4. (**To hand in**) Show that there is a smooth vector field on S^2 which vanishes at exactly one point.
Hint: Try using stereographic projection and consider one of the coordinate vector fields.

Exercise 7.5 (Transition functions and vector bundles)**.** (a) Let $E \xrightarrow{\pi} M$ be a smooth vector bundle of rank k. Suppose that $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M; and for each α we are given a smooth local trivialization

$$\Phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k.$$

For each α, β, let

$$\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$$

[1] be the transition function defined by $\Phi_\alpha \circ \Phi_\beta^{-1}$. Show that the following identity is satisfied for all α, β, γ

$$\tau_{\alpha\beta} \circ \tau_{\beta\gamma} = \tau_{\alpha\gamma}$$

(The juxtaposition on the left-hand side represents matrix multiplication.)

(b) Suppose that $\{U_\alpha\}_{\alpha \in A}$ is an open cover of a smooth manifold M and that for each α, β, we are given smooth maps $\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ satisfying the identity above. Then there exists a smooth vector bundle $E \xrightarrow{\pi} M$ with local trivializations $\Phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ and transition functions $\tau_{\alpha\beta}$

[1]for each point p $\tau_{\alpha\beta}(p)$ is a matrix