Introduction to Differentiable Manifolds	
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Exercise Series 8 - Vector fields and flows	**2021–11–16**

Exercise 8.1. We have seen that given $X \in \mathfrak{X}(M)$ a vector field we have a an $\mathbb{R}$-*linear derivation*

$$X \colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

defined by

$$X(f) \colon M \to \mathbb{R},\ X(f)(p) = X_p f$$

where $X_p \in T_pM$ is the value of the vector field at p and $X_p f$ is given as described in Lecture 3.

(a) Show that the Lie bracket $[X,Y]$ defined by $[X,Y]f = X(Y(f)) - Y(X(f))$ is still a $\mathbb{R}$-linear derivation $[X,Y] \colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$.

(b) Suppose that we have coordinate expressions $X = \sum_i X^i \frac{\partial}{\partial x^i}$, $Y = \sum_j X^j \frac{\partial}{\partial x^j}$. Prove that the Lie bracket is given in coordinates by

$$[X,Y] = \sum_i \sum_j (X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}) \frac{\partial}{\partial x^j}$$

(This exercise is extremely painful, I know! But you do it once in your life and never again.. As ugly as they are, vector fields and their brackets is what allow us to talk about direction derivatives on a manifold and to understand when two direction derivatives commute and when they don't! Notice that the Lie bracket of two coordinatye vector fields in $\mathbb{R}^n$ is always 0)

Exercise 8.2. Let $f : M \to N$ be a smooth map. A vector field $X \in \mathfrak{X}(M)$ is f-**related** to a vector field $Y \in \mathfrak{X}(N)$ if $D_p f(X_p) = Y_{f(p)}$ for all $p \in M$.

(a) X is f-related to Y if and only if $X_p(h \circ f) = Y_{f(p)}(h)$ for all functions $h \in \mathcal{C}^\infty(N, \mathbb{R})$ and all points $p \in M$.

(b) If X is f-related to Y and γ is an integral curve of X, show that $f \circ \gamma$ is an integral curve of Y.

(c) If f is a local diffeo, for every vector field $Y \in \mathfrak{X}(N)$ there exists a unique $X \in \mathfrak{X}(M)$ that is f-related to Y. We denote $f^*Y := X$.
Thus if f is a diffeo, f-relatedness is a bijection from $\mathfrak{X}(M)$ to $\mathfrak{X}(N)$. In this case, if X is f-related to Y, we write $X = f^*Y$ and $Y = f_*X$.

(d) If f is a closed embedding, show that every vector field $X \in \mathfrak{X}(M)$ is f-related to some vector field $Y \in \mathfrak{X}(N)$.
Hint: Construct Y locally, then use partitions of unity.
What happens if f is just an immersion? In this case, find and prove a local version of the fact.

Exercise 8.3. If X is a smooth vector field on a manifold M and $p \in M$ is a point where $X_p \neq 0$, then there exists a chart (U, ϕ) of M defined at p such $X|_U = \frac{\partial}{\partial \phi^0}$. ***Hint:*** It is easier to construct the inverse $\psi = \phi^{-1}$. Use a function of the form $\psi(x) = \Phi_X^{x^0}(f(x^1, \dots, x^{n-1}))$, where $f : U \to M$ is a suitable function defined on an open set $U \subseteq \mathbb{R}^{n-1}$.

Exercise 8.4. (**To hand in**) Compute the flows of the following vector fields.

(a) On the plane $\mathbb{R}^2$, the "angular" vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

(b) A constant vector field X on the torus $\mathbb{T}^n$.

Exercise 8.5. Let X be a $\mathcal{C}^\infty$ tangent vector field on a manifold M, with $k \geq 1$.

(a) For a point $p \in M$ and numbers $s, t \in \mathbb{R}$, show that the equation $\Phi_X^{(s+t)}(p) = \Phi_X^t(\Phi_X^s(p))$ holds if the right-hand side is defined.

(b) We say that X is **complete** if its flow Φ_X is defined over $M \times \mathbb{R}$. Show that a compactly supported vector field is complete. In particular, on a compact manifold, every vector field is complete.
(c) If X is complete, show that the map Φ_X^t is a diffeomorphism $M \to M$.

Exercise 8.6. If X is a complete $\mathcal{C}^\infty$ vector field with $(k \geq 1)$ and $h \in \mathcal{C}^\infty(M, \mathbb{R})$.

(a) Show that the function $X(h) : M \to \mathbb{R}$ that sends $p \mapsto X_p(h)$ is $\mathcal{C}^\infty$.
(b) Show that $X(h) = \frac{\partial}{\partial t}\big|_{t=0} h_t$, where $h_t := (\Phi_X^t)^*(h) = h \circ \Phi_X^t$.
Also show that $X(h_t) = (\Phi_X^t)^*(X(h))$.