Introduction to Differentiable Manifolds	
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Exercise Series 9 - Covector fields, or 1-forms	**2022–11–14**

Exercise 9.1. Show that a covector field ξ on a smooth manifold M is smooth if and only if for any smooth vector field X on M the function $\langle \xi, X\rangle : M \to \mathbb{R}$ defined by $\langle \xi, X\rangle(p) = \xi_p(X_p)$ is smooth.

Exercise 9.2 (Properties of the differential)**.** Let $f, g \in C^\infty(M, \mathbb{R})$.

(a) Prove the formulas: $\mathrm{d}(af + bg) = a\,\mathrm{d}f + b\,\mathrm{d}g$ (where a, b are constants), $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$, $\mathrm{d}\left(\frac{f}{g}\right) = \frac{g\,\mathrm{d}f - f\,\mathrm{d}g}{g^2}$ (on the set where $g \neq 0$)

(b) If $h : \mathbb{R} \to \mathbb{R}$ is a smooth function then $\mathrm{d}(h \circ f) = (h' \circ f)\,\mathrm{d}f$.

(c) If $\mathrm{d}f \equiv 0$, then f is constant on each connected component of M.

Exercise 9.3 (Closed and exact 1-forms)**.** Let M be a smooth manifold, $\omega \in \Omega^1(M)$.

(a) Show that for every $p \in M$ there exists $f \in C^\infty(M)$ such that $\omega|_p = \mathrm{d}f|_p$.
Note that this is only an equality of the covectors at one single point p.

(b) Write $\xi = \sum_i \xi_i\,\mathrm{d}\phi^i$ in some chart (U, ϕ). Show that if ξ is exact, then

$$\frac{\partial}{\partial \phi^j}\xi_i = \frac{\partial}{\partial \phi^i}\xi_j \quad \text{on } U. \tag{1}$$

(c) Use the preceding fact to write down a 1-form which is not exact.

Remark: A 1-form that satisfies (1) for all charts (U, ϕ) is called ***closed****. We have just proved that closedness is a necessary condition for exactness. However, it is not always sufficient. The topology of M comes into play: e.g. on a convex subset of $\mathbb{R}^n$ any closed 1-form is exact. But on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ we can construct a closed 1-form that is not exact.*

Exercise 9.4 (A closed 1-form that is not exact)**.** Let $M = \mathbb{R}^2 \setminus \{0\}$. Let $\omega \in \Omega^1(M)$ be given by

$$\omega = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}.$$

Compute the integral of ω along the curve

$$\gamma : [0, 2\pi] \to M : t \mapsto (\cos t, \sin t).$$

Conclude that ω is not exact.

Exercise 9.5. Let (x, y) be the standard coordinates on $\mathbb{R}^2$ and let (r, φ) be the polar coordinates.

(a) Express $\mathrm{d}x$ and $\mathrm{d}y$ in terms of $\mathrm{d}r$ and $\mathrm{d}\varphi$ (wherever the latter are defined).

(b) Let $G : \mathbb{R}^2 \to \mathbb{R}$, $G(x, y) = x^2 + y^2$. Let t be the standard coordinate on $\mathbb{R}$. Compute $G^*(\mathrm{d}t)$.

Exercise 9.6 (Line integrals)**.** .

(a) Let M be a smooth manifold, $\gamma : I = [a, b] \to M$ a smooth curve and let $\xi \in \Omega^1(M)$. Denote by t the standard coordinate on $\mathbb{R}$. Show that $\int_\gamma \xi = \int_I \gamma^*\xi$.

(b) (Change of variables for 1-forms) Show that if $\sigma : I \to J$ is a positive (i.e. order preserving) diffeo between two intervals $I = [a, b]$, $J = [c, d]$, then $\int_I \sigma^*\theta = \int_J \theta$ for any 1-form $\theta \in \Omega^1(J)$.
Hint: Compute the derivatives of the functions $F(s) = \int_a^s \sigma^*\theta$ and $G(t) = \int_c^t \theta$.
What happens if σ is a negative (i.e. order reversing) diffeo ?

(c) (Reparametrization invariance of curve integrals) If two $\mathcal{C}^1$ curves $\gamma : J \to M$, $\beta : I \to M$ are equivalent as oriented curves, in the sense that β is a positive reparametrization of γ (i.e. $\beta = \gamma \circ \sigma$, where $\sigma : I \to J$ is a positive diffeo), then $\int_\gamma \xi = \int_\beta \xi$ for any 1-form $\xi \in \Omega^1(M)$. Prove this using the definition via pullback.