Introduction to Differentiable Manifolds

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Exercise Series 10 - Manifolds with boundary, line integrals and Tensors 2022–11–22

Manifolds with boundary

Exercise 10.1. Let M be a smooth n-dimensional manifold with boundary, prove that T_pM is an $n-$dimensional real vector space:

(a) First prove that $T_a\mathbb{H}^n \xrightarrow{d\iota} T_a\mathbb{R}^n$ is an isomorphism (using the fact that if f is a smooth function on $\mathbb{H}^n$ then there exists an extension $\widetilde{f}$ to a smooth function on all $\mathbb{R}^n$; look back at Exercise sheet 2 and 3)

(b) As we did in the case of smooth manifolds without boundary, prove that $T_pU \cong T_pM$ for each open U again using the Extension Lemma and then use smooth charts (Here, once again, remember what a smooth chart means in the case of a manifold with boundary)

Line integrals

Exercise 10.2. Let $M = \mathbb{R}^2 \setminus 0$ consider the 1-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2}.$$

Let $\gamma\colon [0, 2\pi] \to M$ the smooth curve defined by $t \mapsto (cost, sint)$.

(a) Compute the integral of ω along γ

(b) Prove that omega is not exact, i.e. is not of the form dh for $h \in C^\infty(M)$

Exercise 10.3 (to hand in). Consider the following 1-form on $M = \mathbb{R}^3$:

$$\omega = \frac{-4zdx}{(x^2+1)^2} + \frac{2ydy}{y^2+1} + \frac{2xdz}{x^2+1}$$

(a) Set up and compute the line integral of ω along the line going from $(0, 0, 0)$ to $(1, 1, 1)$

(b) Consider the smooth map $\Psi\colon W \to \mathbb{R}^3$ given by $(r, \varphi, \theta) \in W := \mathbb{R}^+ \times (0, 2\pi) \times (0, \pi)$:

$$\Psi(r, \varphi, \theta) = (r\cos\varphi \sin\theta, r\sin\varphi\sin\theta, r\cos\theta) \in \mathbb{R}^3.$$

Compute $\Psi^*\omega$

Exercise 10.4. On the plane $\mathbb{R}^2$ with the standard coordinates (x, y) consider the 1-form $\theta = x\,\mathrm{d}y$. Compute the integral of θ along each side of the square $[1, 2] \times [3, 4]$, with each of the two orientations. (There are 8 numbers to compute.)

Tensors

Exercise 10.5. Let $\mathcal{B} = (E_i)_i$ and $\widetilde{\mathcal{B}} = (\widetilde{E}_j)_j$ be two bases of a vector space $V \simeq \mathbb{R}^n$, and let $\mathcal{B}^* = (\varepsilon^i)_i$ and $\widetilde{\mathcal{B}}^* = (\widetilde{\varepsilon}^j)_j$ be the respective dual bases. Note that a tensor $T \in \operatorname{Ten}^k V$ can be written as

$$T = \sum_{i_0,\ldots,i_{k-1}} T_{i_0,\ldots,i_{k-1}}\, \varepsilon^{i_0} \otimes \cdots \otimes \varepsilon^{i_{k-1}} \quad \text{or as} \quad T = \sum_{j_0,\ldots,j_{k-1}} \widetilde{T}_{j_0,\ldots,j_{k-1}}\, \widetilde{\varepsilon}^{j_0} \otimes \cdots \otimes \widetilde{\varepsilon}^{j_{k-1}}.$$

Find the transformation law that expresses the coefficients $\widetilde{T}_{j_0,\ldots,j_{k-1}}$ in terms of the coefficients $T_{i_0,\ldots,i_{k-1}}$.

Exercise 10.6 (Alternating covariant tensors)**.** Let V be a finite-dimensional real vector space. Let $T \in \operatorname{Ten}^k V$. Suppose that with respect to some basis ε^i of V^*

$$T = \sum_{1 \leq i_1, \ldots, i_k < n} T_{i_1 \cdots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}.$$

Show that T is alternating iff:

(a) for all $\sigma \in S_k$: $T_{i_{\sigma(1)} \cdots i_{\sigma(k)}} = \operatorname{sgn}(\sigma)\, T_{i_1 \cdots i_k}$.
(b) $T(\ldots, X_s, \ldots, X_t, \ldots) = -T(\ldots, X_s, \ldots, X_t, \ldots)$