Introduction to Differentiable Manifolds

EPFL – Fall 2022 F. Carocci, M. Cossarini

Exercise Series 11 - Alternating tensors, differential forms 2022–11–28

Exercise 11.1. Let $T \in \mathrm{Ten}^k(V)$ be a tensor on a real vector space V, and let $T_I = T(E_I)$, defined for any multi-index $I = (i_0, \dots, i_{k-1})$ be the coefficients of T w.r.t. some base $(E_i)_i$ of V. Show that $(\sigma T)_I = T_{\sigma^* I}$ for any permutation $\sigma \in S_k$

Exercise 11.2. Let M be a smooth manifold and let ω be a differential k-form on M. We say that ω is smooth at some point $p \in M$ if the component functions of ω w.r.t. some chart φ (that is defined at p) are $\mathcal{C}^r$ at p. Show that this does not depend on which chart φ we use.

Exercise 11.3. The goal of this exercise is to show that the wedge product of alternating covariant tensors in a real vector space V is associative.

(a) If a tensor $T \in \mathrm{Ten}^k V$ is alternating, show that $A(T) = k!\, T$.
(b) For two tensors $S \in \mathrm{Ten}^k V$, $T \in \mathrm{Ten}^\ell V$, show that

$$A(A(S) \otimes T) = k!\, A(S \otimes T)$$
$$A(S \otimes A(T)) = \ell!\, A(S \otimes T).$$

(c) Show that the wedge product of alternating tensors $S \in \mathrm{Alt}^k V$, $T \in \mathrm{Alt}^\ell V$, $R \in \mathrm{Alt}^m V$ is associative:

$$S \wedge (T \wedge R) = S \wedge T \wedge R = (S \wedge T) \wedge R.$$

Exercise 11.4. Prove that for each k

$$\bigwedge^k T^*M := \coprod_{p \in M} \bigwedge^k T_p^* M$$

are vector bundles over M of rank $\binom{n}{k}$.

Hint: Show that given a coordinate chart (U, φ) of M we have a trivialization of $\bigwedge^k T^*M$; given two such trivialization compute the transition function in terms of the change coordinates to conclude.

Exercise 11.5 (to hand in). For a point $p \in \mathbb{R}^3$ and vectors $v, w \in \mathrm{T}_p\mathbb{R}^3 \equiv \mathbb{R}^3$ we define $\omega|_p(v, w) := \det(p \mid v \mid w)$. Show that ω is a smooth differential 2-form on $\mathbb{R}^3$, and express ω as a linear combination of the elementary alternating 2-forms determined by the standard coordinate chart (x^0, x^1, x^2).

Exercise 11.6 (Some properties of the pullback of differential forms)**.** For $F : M \to N$ a smooth map between smooth manifolds, $\omega \in \Omega^k(N)$, $\beta \in \Omega^\ell(N)$ we have:

(a) $F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta)$.
(b) In any coordinate chart y^i on N,

$$F^* \left(\sum_{\substack{I=(i_0,\dots,i_{k-1}) \\ 0 \le i_0,\dots,i_{k-1} < n}} \omega_I \, \mathrm{d}y^I \right) = \sum_{\substack{I=(i_0,\dots,i_{k-1}) \\ 0 \le i_0,\dots,i_{k-1} < n}} (\omega_I \circ F)\, \mathrm{d}(y^{i_0} \circ F) \wedge \cdots \wedge \mathrm{d}(y^{i_{k-1}} \circ F).$$

(c) $F^*(\omega) \in \Omega^k(M)$.

Exercise 11.7. Define a 2-form ω on $\mathbb{R}^3$ by

$$\omega = x \; \mathrm{d}y \wedge \mathrm{d}z + y \; \mathrm{d}z \wedge \mathrm{d}x + z \; \mathrm{d}x \wedge \mathrm{d}y.$$

(a) Compute ω in spherical coordinates.
(b) Show that $\omega|_{\mathbb{S}^2}$ is nowhere 0.
(c) Compute the exterior derivative $\mathrm{d}\omega$.