Introduction to Differentiable Manifolds	
EPFL – Fall 2022	F. Carocci, M. Cossarini
Exercise Series E - Exam training	**2022–12–20**

Notes: the following rules will apply during the exam

- You are not allowed to use any electronic devices during the exam.
- You can have a single one-sided handwritten sheet of paper with your notes.
- All questions are worth the same number of points.
- Incomplete solutions may be awarded some points.

Exercise E.1. (a) Define the notion of a smooth map between two smooth manifolds.
(b) State the constant rank theorem.
(c) Explain what are immersions, embeddings and submersions.
(d) Give an example of an immersion which is not an embedding (justify your answer).
(e) Prove that all submersions are open maps.

Exercise E.2. Let M be a topological n-manifold with two smooth atlases $\mathcal{A}_0$, $\mathcal{A}_1$, defining two smooth manifolds $M_i = (M, \overline{\mathcal{A}_i})$. Show that the atlases $\mathcal{A}_0$, $\mathcal{A}_1$ are equivalent if and only if the following condition holds: for every smooth manifold N and every continuous function $f : M \to N$, the function f is smooth as a function $M_0 \to N$ if and only if it is smooth as a function $M_1 \to N$.
Hint: Consider first the case where each atlas $\mathcal{A}_i$ consists of a single chart.

Exercise E.3. For which values of $a \in \mathbb{R}$ is the set

$$M_a = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x(x-1)(x-a)\}$$

a smooth embedded curve ? For which values is M_a an immersed curve ?
Note: A smooth embedded (resp. immersed) curve in $\mathbb{R}^2$ is the image of a smooth embedding (resp. immersion) $N \to \mathbb{R}^2$, where N is a smooth 1-manifold.
Hint: At some moment the curve $\gamma(t) = (t^2, t(t^2-1))$ may be useful.

Exercise E.4. Show that every smooth submersion $f : M \to N$ between manifolds without boundary is an open map, i.e. for every open set $U \subseteq M$, the image $f(U)$ is an open subset of N.

Exercise E.5. If X is a vector field on a manifold M, for each differentiable function $h : M \to \mathbb{R}$ we denote $X(h)$ the function $M \to \mathbb{R}$ that maps $p \mapsto X|_p(h)$, where $X|_p$ acts on h as a derivation.

(a) Show that if X is and h are smooth, then $X(h)$ is smooth.
(b) Show that if X and Y are smooth vector fields on M, then there exists a unique vector field Z such that

$$Z(h) = X(Y(h)) - Y(X(h)) \quad \forall h \in \mathcal{C}^\infty(M, \mathbb{R}).$$

Hint: Consider first the case where M is covered by a single chart.
(c) Show that if X, Y are smooth, then so Z.

Exercise E.6. Consider the 1-form $\theta = x\,\mathrm{d}y - y\,\mathrm{d}x$ on the circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Show that every smooth 1-form α on $\mathbb{S}^1$ is a smooth multiple of θ. That is, $\alpha = h\,\theta$ for some smooth function $h : \mathbb{S}^1 \to \mathbb{R}$.

Exercise E.7. Let M be a smooth n-manifold. Show that its tangent bundle $\mathrm{T}M$, with its natural smooth structure, is an orientable smooth $2n$-manifold.