**Exercise H.1.** Show that the **projective space** $\mathbb{P}^n$, defined as the quotient of $\mathbb{R}^{n+1}\setminus\{0\}$ by the equivalence relation $x \sim y$ iff $x = \lambda\, y$ for some $\lambda \in \mathbb{R}\setminus\{0\}$, is a smooth $n$-manifold with atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i=0,\dots,n}$ given by

$$U_i := \{[x] \in \mathbb{P}^n \mid x_i \neq 0\}, \qquad \varphi_i([x]) = \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right),$$

where $[x] \in \mathbb{P}^n$ denotes the equivalence class of a point $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}\setminus\{0\}$.

*Solution.* We will prove several facts

(a) *The quotient map $\pi : \mathbb{R}^{n+1}_{\neq 0} \to \mathbb{P}^n$ is open.* Let $U \subseteq \mathbb{R}^{n+1}_{\neq 0}$ be an open set. To see that $\pi(U)$ is open in the quotient topology, we verify that its preimage $\pi^{-1}(\pi(U)) = \bigcup_{\lambda\neq 0} \lambda U$ is open, being a union of open sets $\lambda U$.

(b) *$\mathbb{P}^n$ is second countable.* Cleary $\mathbb{R}^{n+1}_{\neq 0}$ is second countable, being a subspace of the countable space $\mathbb{R}^{n+1}$. Let $(W_j)_{j\in\mathbb{N}}$ be a countable topological basis for $\mathbb{R}^{n+1}_{\neq 0}$. Then $(\pi(W_j))_{j\in\mathbb{N}}$ is a countable basis for $\mathbb{P}^n$, being the image of a topological basis by a surjective open map.

(c) *$\mathbb{P}^n$ is locally homeomorphic to $\mathbb{R}^n$.* To see that $U_i$ is open in the quotient topology, we verify that its preimage $\pi^{-1}(U_i)$ is open in $\mathbb{R}^{n+1}_{\neq 0}$. And indeed, its preimage is the set

$$V_i = \{x \in \mathbb{R}^{n+1}_{\neq 0} : x_i \neq 0\},$$

which is open. To see that $\varphi_i$ is a bijection, let's find its inverse function. A direct calculation provides us with the formula

$$\varphi_i^{-1}(x_0, \dots, x_{n-1}) = [x_0, \dots, x_{i-1}, 1, x_i, \dots, x_n].$$

This formula also shows that $\varphi_i^{-1}$ is continuous. Finally, to see that $\varphi_i$ itself is continuous, it suffices to note that the composite map

$$\widetilde{\varphi}_i = \varphi_i \circ \pi|_{V_i}^{U_i} : V_i \to \mathbb{R}^n$$

is continuous. (Here we are using the universal property of the quotient. The map $\pi|_{V_i}^{U_i}$ is a quotient map because it is surjective and open.) And indeed, the map

$$\widetilde{\varphi}_i(x_0, \dots, x_n) = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

is continuous. Thus $\varphi_i$ is a homeomorphism between the open set $U_i \subseteq \mathbb{P}^n$ and $\mathbb{R}^n$.

(d) *$\mathbb{P}^n$ is Hausdorff.* Let $[x]$, $[y]$ be two distinct points of $\mathbb{P}^n$. We will show there are disjoint open neighboorhoods $U, V$ of $x, y$ in $\mathbb{R}^{n+1}_{\neq 0}$ that are saturated by the equivalence relation $\sim$. Then it follows that $\pi(U)$, $\pi(V)$ are disjoint respective neighorhoods of $[x]$, $[y]$ in $\mathbb{P}^n$.

We consider two cases. The first case is when both points $x, y$ have a nonzero coordinate at the same place, i.e. $x_i, y_i \neq 0$ for some $i$. Then the points $[x], [y]$ are contained in the open subset $U_i$, which is homeomorphic to $\mathbb{R}^n$, hence Hausdorff. Thus there are disjoint open neighborhoods $V, W$ of $[x], [y]$ in $U_i$, and these sets are also open in $\mathbb{P}^n$.

The remaining case is when there is no $i$ such that $x_i, y_i \neq 0$. In this case let $i$, $j$ such that $x_i \neq 0$ (hence $y_i = 0$) and $y_j \neq 0$ (hence $x_j = 0$). Then we have in $\mathbb{R}^{n+1}_{\neq 0}$ the saturated open sets

$$V = \{z \in \mathbb{R}^{n+1}_{\neq 0} : |z_i| > |z_j|\}$$

$$W = \{z \in \mathbb{R}^{n+1}_{\neq 0} : |z_j| > |z_i|\}$$

which are disjoint neighborhoods of $x$ and $y$ respectively.

Alternative way of showing that an open quotient is Hausdorff: Show that the relation $R \subseteq (\mathbb{R}^{n+1} \setminus \{0\})^2$ consisting of the pairs $(z, \lambda z)$ is closed...

(e) Smooth structure.

Convention: All indices $i, j, k$ are in the set $n' = n+1 = \{0, \dots, n\}$.
For each $i$ we have a chart $\phi_i : U_i \to \mathbb{R}^{n' \setminus \{k\}} \equiv \mathbb{R}^n$, given by

$$U_i = \{[x] \in \mathbb{P}^n : x^i \neq 0\} \subseteq \mathbb{P}^n,$$

$$\phi_i : [x] \mapsto \left(\frac{x_j}{x_i}\right)_{j \neq i}.$$

Its inverse is $\phi_i^{-1} : (y^j)_{j \neq i} \mapsto [x^j]_j$ where $x^j := y^j$ if $j \neq i$ and $x^i := 1$.
The nontrivial transition functions are $\phi_k \circ \phi_i^{-1}$, with $k \neq i$, defined on

$$\phi_i(U_k) = \{x \in \mathbb{R}^{n' \setminus \{i\})} : x_k \neq 0\}$$

by the formula

$$\phi_k \circ \phi_i^{-1} : y \mapsto (\frac{x^j}{y_k})_{j \neq k},$$

where the $x^j$ is defined as above: $x^j = y^j$ if $j \neq i$, $x^i = 1$. The transition maps are smooth, therefore the atlas is smooth.

□

**Exercise H.2 (to hand in).** Prove the following

(a) Let $c : M \to N$ the constant map between two smooth manifolds; $c$ is smooth
(b) Every smooth chart $\varphi : U \to \varphi(U)$ of $M$ is a diffeomorphism; here $U$ and $\varphi(U)$ are given the open subspace smooth structure defined in Exercise 1.4.
(c) The composite $g \circ f$ of two smooth maps $f : M \to N$, $g : N \to P$ is smooth map.
(d) Show that the quotient map $\pi : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ is a smooth map of manifolds where on $R\mathbb{P}^n$ we considered the smooth structure defined in Exercise 1.7.

**Exercise H.3.** (**To hand in**) Consider the inclusion $\iota : S^2 \to \mathbb{R}^3$, where we endow both spaces with the standard smooth structure. Let $p \in S^2$. What is the image of $D_p\iota : T_pS^2 \to T_p\mathbb{R}^3$? (Identify $T_p\mathbb{R}^3$ with $\mathbb{R}^3$ in the standard way, i.e. $e_i \mapsto \frac{\partial}{\partial x^i}|_p$) So the result should be the equation for a plane in $\mathbb{R}^3$.)

*Solution.* We will do the computations for a point $p = (p^0, p^1, p^2) \in S^2$ such that $p^2 > 0$. (The other cases are similar.)

This point $p$ is contained in the open set $U = U_2^+ = \{x \in S^2 : x^2 > 0\}$, which is the domain of the chart $\varphi = \varphi_2^+ : (x^0, x^1, x^2) \mapsto (x^0, x^1)$.

The local expression of the inclusion map $\iota : S^2 \to \mathbb{R}^3$ with respect to the charts $\varphi$ and $\mathrm{id}_{\mathbb{R}}^3$ is the map $\widetilde{\iota} : (x^0, x^1) \mapsto (x^0, x^1, \sqrt{1-(x^0)^2-(x^1)^2})$, whose Jacobian matrix at the point $x = (x_0, x_1) = \iota^{-1}(p)$ is

$$J_x\widetilde{\iota} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x^0}{\sqrt{1-(x^0)^2-(x^1)^2}} & \frac{-x^1}{\sqrt{1-(x^0)^2-(x^1)^2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-p^0}{p^3} & \frac{-p^1}{p^3} \end{pmatrix}.$$

This implies that the differential $\mathrm{D}_p\iota$ sends the vectors $\left.\frac{\partial}{\partial\varphi^0}\right|_p$, $\left.\frac{\partial}{\partial\varphi^1}\right|_p$ (which constitute a basis of $T_pS^2$) to the vectors

$$\mathrm{D}_p\iota\left(\left.\frac{\partial}{\partial\varphi^0}\right|_p\right) = \left.\frac{\partial}{\partial x^0}\right|_p - \frac{p^0}{p^3}\left.\frac{\partial}{\partial x^2}\right|_p \cong \left.\left(1, 0, -\frac{p^0}{p^3}\right)\right|_p,$$

$$\mathrm{D}_p\iota\left(\left.\frac{\partial}{\partial\varphi^1}\right|_p\right) = \left.\frac{\partial}{\partial x^1}\right|_p - \frac{p^1}{p^3}\left.\frac{\partial}{\partial x^2}\right|_p \cong \left.\left(0, 1, -\frac{p^1}{p^3}\right)\right|_p,$$

Therefore, the image of $D_p\iota$ is the vector space spanned by these two image vectors, which coincides with the orthogonal space of $p$,

$$S = p^{\perp}|_p = \{v|_p : v \in \mathbb{R}^3 \text{ such that } \langle p, v\rangle = 0\}.$$

$\square$

**Exercise H.4 (To hand in).** Let $f : M \to N$ be an injective immersion of smooth manifolds. Show that there exists a closed embedding $M \to N \times \mathbb{R}$.

***Hint:*** Recall that there exists a proper map $g : M \to \mathbb{R}$ (Exercise 3.2)

*Solution.* The map $h : M \to N \times \mathbb{R} : x \mapsto (f(x), g(x))$ is an immersion and is proper, hence it is a closed embedding.

Proof that $h$ is proper: Let $K \subseteq N \times \mathbb{R}$ a compact set. Note that $K$ is closed in $N$ since it's a compact subset of a Hausdorff space. It follows that $h^{-1}(K)$ is closed. In addition $h^{-1}(K)$ is contained in the compact set $g^{-1}(\pi_1(K))$, where $\pi_1 : N \times \mathbb{R} \to \mathbb{R}$ is the projection. Therefore $h^{-1}(K)$ is compact. This proves that $h$ is proper, hence closed. Since in addition it is injective, it's a closed topological embedding.

Proof that $h$ is an immersion: for each nonzero vector $v \in \mathrm{T}_pM$, the vector $\mathrm{T}_ph(v) = (\mathrm{T}_pf(v), \mathrm{T}_pg(v))$ is nonzero because its first component $\mathrm{T}_pf(v)$ is nonzero. $\square$

**Exercise H.5 (To hand in).** Show that the following subgroups of $GL_n(\mathbb{R})$ are closed submanifolds. Compute their dimension and their tangent space at the identity.

(a) The *special linear group* $\mathrm{SL}_n(\mathbb{R})$, consisting of matrices with determinant equal to 1.

*Solution.* The determinant function $\det : M_n \to \mathbb{R}$ is continuous, which implies that the preimage of a closed (resp. open) set is a closed (resp. open) set. We have already used this to show that the *general linear group* $GL_n = \det^{-1}(\mathbb{R}_{\neq 0})$ is open in $M_n$. And now we can use it to show that the *special linear group* $SL_n = \det^{-1}(1)$ is a closed subset of $M_n$. (And since $SL_n$ is contained in $GL_n$, it is also closed in $GL_n$).

To show that $SL_n$ is a submanifold we use the regular preimage theorem. We apply the theorem to the determinant map $\det : M_n \to \mathbb{R}$, which is a smooth map (by a previous exercise).

To apply the theorem we have to show that 1 is a regular value of det. Thus we have to show that the linear transformation

$$\mathrm{D}_A \det : \mathrm{T}_A M_n \equiv \mathbb{R}^{n^2} \longrightarrow \mathrm{T}_{\det(A)}\mathbb{R} \equiv \mathbb{R}$$

is surjective for all points $A \in SL_n$. Since the codomain of this linear transformation has dimension 1, we have two possibilities: either the transformation is surjective (if it has rank 1) or it is null (if it has rank 0). Thus it suffices to show that the transformation $\mathrm{D}_A \det$ is not null. We have already computed the differential

$$\mathrm{D}_A \det(X) = \det(A)\,\mathrm{tr}(A^{-1}X)$$

Putting $X := A$ we get

$$\mathrm{D}_A \det(X) = \det(A)\,\mathrm{tr}(I_n) = n$$

This implies that $\mathrm{D}_A \det$ is surjective for every $A \in SL_n$. Therefore $SL_n = \det^{-1}(1)$ is an embedded submanifold of $M_n$ of dimension

$$\dim(SL_n) = \dim(M_n) - \dim(\mathbb{R}) = n^2 - 1.$$

Finally, the regular preimage theorem also tells us that the tangent space of $SL_n$ at any point $A \in SL_n$ is

$$\mathrm{T}_A(SL_n) = \mathrm{Ker}(\mathrm{D}_A \det) = \{X \in M_n \mid \mathrm{tr}(A^{-1}X) = 0\}$$

In particular,

$$\mathrm{T}_{I_n}(SL_n) = \{X \in M_n \mid \mathrm{tr}(X) = 0\}.$$

We have shown that the tangent space of an open subset is equal to the tangent of the whole manifold which contains this open subset, so here $T_pGL_n(\mathbb{R}) = T_pM_n(\mathbb{R})$ since $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R}_{\neq 0})$ is an open subset of $M_n(\mathbb{R})$. This implies that

$$\dim(SL_n(\mathbb{R})) = \dim GL_n(\mathbb{R}) - 1 = n^2 - 1.$$

□

(b) The *orthogonal group* $O_n(\mathbb{R})$, consiting of the orthogonal matrices $A$ (which satisfy $A^\top A = I_n$).

***Hint:*** Consider the map $f : M_n \to M_n^{sym}$ that sends $A \mapsto A^\top A$, there $M_n^{sym}$ is the vector space of *symmetric* $n \times n$ matrices.

*Solution.* Note that $f^{-1}(I_n) = O_n$. To apply the regular preimage theorem we have to verify that $I_n$ is a regular value of $f$. Thus we have to show that for each point $A \in O_n$, the linear transformation

$$\mathrm{D}_A f : \mathrm{T}M_n \equiv M_n \longrightarrow \mathrm{T}M_n^{sym} \equiv M_n^{sym}$$

is surjective. Note that

$$\begin{aligned} \mathrm{D}_A f(X) &= A^\top X + X^\top A \\ &= A^\top X + (A^\top X)^\top. \end{aligned}$$

Let $Y \in M_n^{sym}$ be an antisymmetric matrix. Let us find some $X \in M_n$ such that $\mathrm{D}_A f(X) = Y$. We can write $Y = \frac{1}{2}Y + \frac{1}{2}Y^\top$, thus it suffices to find $X \in M_n$ such that $A^\top X = \frac{1}{2}Y$. We put simply $X = (A^\top)^{-1}\frac{1}{2}Y = \frac{1}{2}AY$. This finishes the proof that $I_n$ is a regular value of $f$. Therefore, by the regular preimage theorem, the set $O_n = f^{-1}(I_n)$ is a closed embedded submanifold of $M_n$ of dimension

$$\dim(O_n) = \dim(M_n) - \dim(M_n^{sym}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Its tangent space at any point $A \in O_n$ is

$$D_A O_n = \mathrm{Ker}\ \mathrm{D}_A f = \{X \in \mathrm{T}M_n \mid A^\top X + X^\top A = 0\}$$

In particular, its tangent space at the identity matrix is

$$D_{I_n}(O_n) = \{X \in M_n \mid X + X^\top = 0\},$$

that is, the space of antisymmetric matrices. □

**Exercise H.6 (To hand in).** (a) Show that the map $f : \mathbb{P}^2 \to \mathbb{R}^3$ defined by

$$f([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy).$$

is smooth, and has injective differential except at 6 points.

(b) Show that the map $g : \mathbb{P}^2 \to \mathbb{R}^4$ defined by

$$g([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy, x^2 - z^2)$$

is a smooth embedding.

**Exercise H.7.** Show that there is a smooth vector field on $S^2$ which vanishes at exactly one point.

***Hint:*** Try using stereographic projection and consider one of the coordinate vector fields.

*Solution.* Recall that

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

Let us denote $(u, v)$ the stereographic coordinates relative to the projection from the north pole $N = (0, 0, 1)$, that is, the map

$$\begin{aligned} \phi : \mathbb{S}^2 \setminus \{N\} &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (u, v) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right). \end{aligned}$$

(Note that we use the letters $u, v$ to denote real numbers but also to denote the component functions $\phi^0$, $\phi^1$ of the chart $\phi$, which are functions $\mathbb{S}^2 \to \mathbb{R}$.)

Similarly, denote $(\overline{u}, \overline{v})$ the stereographic coordinates relative to the projection from the south pole $S = (0, 0, -1)$, which is the map

$$\begin{aligned} \psi : \mathbb{S}^2 \setminus \{S\} &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (\bar{u}, \bar{v}) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right) \end{aligned}$$

The transition function $\psi \circ \phi^{-1}(u, v)$ is obtained after some computation:

$$(\overline{u}, \overline{v}) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$$

For this we use the inverse of the north stereographic projection which is

$$x = \frac{2u}{1+u^2+v^2} \qquad y = \frac{2v}{1+u^2+v^2} \qquad z = \frac{-1+u^2+v^2}{1+u^2+v^2}.$$

Let $X = \frac{\partial}{\partial \phi^0} = \frac{\partial}{\partial u}$ be the first coordinate vector field of the chart $\phi$. This vector field $X$ is a non-vanishing smooth vector field defined on $\mathbb{S}^2 \setminus \{N\}$. (Its component functions w.r.t. $\phi$ are just the constant functions 1 and 0; therefore $X$ is smooth.) The important step is to show that $X$ extends to a smooth vector field defined on the whole sphere.

For this we compute the component functions w.r.t. $\psi$ on the intersection of the two charts, i.e. on $\mathbb{S}^2 \setminus \{N, S\}$:

$$\begin{aligned} X &= \frac{\partial \psi^0}{\partial \phi^0} \frac{\partial}{\partial \psi^1} + \frac{\partial \psi^1}{\partial \phi^0} \frac{\partial}{\partial \psi^1} \\ &= \frac{\partial \overline{u}}{\partial u} \frac{\partial}{\partial \overline{u}} + \frac{\partial \overline{v}}{\partial u} \frac{\partial}{\partial \overline{v}} \\ &= \frac{v^2-u^2}{(u^2+v^2)^2} \frac{\partial}{\partial \overline{u}} + \frac{-2uv}{(u^2+v^2)^2} \frac{\partial}{\partial \overline{v}} \\ &= (\overline{v}^2 - \overline{u}^2) \frac{\partial}{\partial \overline{u}} - 2\overline{uv} \frac{\partial}{\partial \overline{v}} \end{aligned}$$

From this we see that $X$ can be extended to a smooth vector field $X$ on the whole sphere by setting its value on the north pole to zero., i.e.

$$X|_p = \begin{cases} \frac{\partial}{\partial u}\big|_p & \text{if } p \in \mathbb{S}^2 \setminus \{N\} \\ 0 & \text{if } p = N. \end{cases}$$

Then on $\mathbb{S}^2 \setminus \{S\}$

$$X = (\overline{v}^2 - \overline{u}^2) \frac{\partial}{\partial \psi^1} - 2\overline{uv} \frac{\partial}{\partial \psi^2} \quad \text{on } \mathbb{S}^2 \setminus \{S\}$$

so the component functions w.r.t. $\psi$ are smooth as functions on $\mathbb{S}^2 \setminus \{S\}$. By construction the component functions of $X$ w.r.t. $\phi$ are smooth as functions on $\mathbb{S}^2 \setminus \{N\}$.

*Remark:* We could also have used the reverse of the north stereographic projection to the sphere

$$x = \frac{2u}{1+u^2+v^2} \qquad y = \frac{2v}{1+u^2+v^2} \qquad z = \frac{-1+u^2+v^2}{1+u^2+v^2}$$

to express $\frac{\partial}{\partial \phi^1}$ in cartesian coordinates (a priori the expression below is only defined for $(x, y, z) \in \mathbb{S}^2 \setminus \{N\}$)

$$\begin{aligned} (\varphi_N^{-1})_* \frac{\partial}{\partial u} &= \frac{2(1-u^2+v^2)}{(1+u^2+v^2)^2} \frac{\partial}{\partial x} + \frac{4uv}{(1+u^2+v^2)^2} \frac{\partial}{\partial y} + \frac{4u}{(1+u^2+v^2)^2} \frac{\partial}{\partial z} \\ &= (1-z-x^2) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + x(1-z) \frac{\partial}{\partial z} \end{aligned}$$

and argue that this extends to a smooth vector field on the sphere. □

**Exercise H.8.** (**To hand in**) Compute the flows of the following vector fields.

(a) On the plane $\mathbb{R}^2$, the "angular" vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

*Solution.* The integral curves are of the form $\gamma(t) = \begin{pmatrix} r\cos(t-t_0) \\ r\sin(t-t_0) \end{pmatrix}$, with $t_0 \in \mathbb{R}$ and $r \geq 0$. We can rewrite them as

$$\begin{aligned} \gamma(t) &= \begin{pmatrix} r\cos(t-t_0) \\ r\sin(t-t_0) \end{pmatrix} \\ &= \begin{pmatrix} r\cos(t)\cos(t_0) + r\sin(t)\sin(t_0) \\ r\sin(t)\cos(t_0) - r\cos(t)\sin(t_0) \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} r\cos(t_0) \\ -r\sin(t_0) \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \end{aligned}$$

where $(x_0, y_0) = \gamma(0)$. Thus the flow is $\Phi_X^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, defined for all points $(x_0, y_0) \in \mathbb{R}^2$ and all $t \in \mathbb{R}$. □

(b) A constant vector field $X$ on the torus $\mathbb{T}^n$. (What is a constant vector field on the torus?)

*Solution.* Note first that we have an identification $\mathrm{T}_{[p]}\mathbb{T}^n \equiv \mathbb{R}^n$ for all points $[p] = \pi(p) \in \mathbb{T}^n$, where $p \in \mathbb{R}^n$ and $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is the quotient map. This identification is the linear transformation $D_p\pi$, which is an isomorphism from $D_p\mathbb{R}^n \equiv \mathbb{R}^n$ to $\mathrm{T}_{[p]}\mathbb{T}^n$. This identification $D_p\pi : \mathbb{R}^n \to D_p\mathbb{T}^n$ is independent of which preimage we choose for $[p]$, since if $p'$ is another preimage and $\tau$ is the translation of $\mathbb{R}^n$ that maps $p \mapsto p'$, then $\pi(x) = \pi \circ \tau$, and therefore

$$D_p\pi = \mathrm{T}_{p'}\pi \circ \mathrm{T}_p\tau \equiv \mathrm{T}_{p'}\pi$$

since $D_p\tau \equiv \mathrm{id}_{\mathbb{R}^n}$.

Thus we can talk about a constant vector field $X$ on $\mathbb{T}^n$. This means that

$$X_{[p]} = a \qquad \text{for all } p \in \mathbb{R}^n$$

for some fixed $a \in \mathbb{R}^n$.

Let $\widehat{X} = \pi^*X$ be the vector field on $\mathbb{R}^n$ given by the similar formula $\widehat{X}_p = a$ for all $p \in \mathbb{R}^n$. Note that $\widehat{X}$ is $\pi$-related to $X$, where $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is the quotient map. Therefore $\pi \circ \gamma$ is an integral curve of $X$ if $\gamma$ is an integral curve of $\widehat{X}$.

For any point $p \in \mathbb{R}^n$, the maximal integral curve of $\widehat{X}$ starting at the point $p$ is $\gamma_{\widehat{X},p}(t) = p + at$. Therefore the curve

$$\gamma_{X,[p]}(t) := \pi(\gamma_{\widehat{X},p}(t)) = [p + ta]$$

is an integral curve of $X$. It has initial condition $\gamma_{X,[p]}(0) = [p]$ and it is maximal because it is defined for all $t$.

Therefore the flow of $X$ is $\Phi_X^t[p] = [p+ta]$, which is defined for all points $[p] \in \mathbb{T}^n$ and all $t \in \mathbb{R}$. □

**Exercise H.9 (to hand in).** Consider the following 1-form on $M = \mathbb{R}^3$:

$$\omega = \frac{-4z\,\mathrm{d}x}{(x^2+1)^2} + \frac{2y\,\mathrm{d}y}{y^2+1} + \frac{2x\,\mathrm{d}z}{x^2+1}$$

(a) Set up and compute the line integral of $\omega$ along the line going from $(0,0,0)$ to $(1,1,1)$

*Solution.* This line is parametrized by the curve $\gamma : t \in [0,1] \mapsto \gamma(t) = (t,t,t)$. The velocity vector of this curve is $\gamma'(t) = (1,1,1)$. Therefore the pullback of

$$\begin{aligned}\int_\gamma \omega &= \int_{[0,1]} \gamma^*\omega \\ &= \int_0^1 \left( \frac{-4t}{(t^2+1)^2} + \frac{2t}{t^2+1} + \frac{2t}{t^2+1} \right) \mathrm{d}t \\ &= \int_0^1 \frac{-4t + 4t(t^2+1)}{(t^2+1)^2} \mathrm{d}t \\ &= \int_0^1 \frac{4t^3}{(t^2+1)^2} \mathrm{d}t \\ &= \left[ 2\left( \frac{1}{t^2+1} + \log(t^2+1) \right) \right]_{t=0}^{t=1} \\ &= \log(4) - 1\end{aligned}$$

□

(b) Consider the smooth map $\Psi : W \to \mathbb{R}^3$ given by $(r,\varphi,\theta) \in W := \mathbb{R}^+ \times (0,2\pi) \times (0,\pi)$:
$$\Psi(r,\varphi,\theta) = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta) \in \mathbb{R}^3.$$
Compute $\Psi^*\omega$.

**Exercise H.10 (to hand in ).** For a point $p \in \mathbb{R}^3$ and vectors $v, w \in \mathrm{T}_p\mathbb{R}^3 \equiv \mathbb{R}^3$ we define $\omega|_p(v,w) := \det(p \mid v \mid w)$. Show that $\omega$ is a smooth differential 2-form on $\mathbb{R}^3$, and express $\omega$ as a linear combination of the elementary alternating 2-forms determined by the standard coordinate chart $(x^0, x^1, x^2)$.

*Solution.* For each point $p \in \mathbb{R}^3$, the function $\omega|_p(v,w) = \det(p \mid v \mid w)$ is linear on each of its two variables $v, w \in \mathbb{R}^3$, and also alternating, therefore $\omega$ is a differential form. The elementary covector fields are $\mathrm{d}x^0$, $\mathrm{d}x^1$, $\mathrm{d}x^2$, and the elementary 2-forms are $\mathrm{d}x^0 \wedge \mathrm{d}x^1$, $\mathrm{d}x^1 \wedge \mathrm{d}x^2$ and $\mathrm{d}x^2 \wedge \mathrm{d}x^0$. The calculation

$$\begin{aligned}\omega|_p(v,w) = \det \begin{pmatrix} p^0 & v^0 & w^0 \\ p^1 & v^1 & w^1 \\ p^2 & v^2 & w^2 \end{pmatrix} = \; & p^0(v^1w^2 - v^2w^1) \\ & + p^1(v^2w^0 - v^0w^2) \\ & + p^2(v^0w^1 - v^1w^0)\end{aligned}$$

shows that

$$\begin{aligned}\omega|_p = \; & p^0 \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ & + p^1 \, \mathrm{d}x^2 \wedge \mathrm{d}x^0 \\ & + p^2 \, \mathrm{d}x^0 \wedge \mathrm{d}x^1.\end{aligned}$$

Thus the component functions of $\omega$ are the functions $p \mapsto p^i$ which are smooth. This shows that $\omega$ is a smooth 2-form. □