Exercise Series 2 - Differentiable manifolds and maps **2022–09–27**

Exercise 2.1 (Manifolds from vector spaces)**.** Let V be a vector space and , $|\cdot| : V \to \mathbb{R}$ a norm. Then V is endowed with a distance function and thus has the structure of a topological space. Let $E_1, \dots, E_n$ be a basis for V, then $E\colon \mathbb{R}^n \to V$ defined by $(x_1, \dots, x_n) \to \sum_i x_i E_i$.

(a) Show that (V, E^{-1}) is a chart for V;
(b) Show that given a different base $\widetilde{E}_1, \dots, \widetilde{E}_n$ (V, E^{-1}), $(V, \widetilde{E}^{-1})$ are smoothly compatible.
We say that the collection of charts of this form define the *standard smooth structure* on V.

Use the previous part of the exercise to show that:

(a) The space $M(n \times m, \mathbb{R})$ of $n \times m$ matrices has a natural smooth manifold structure
(b) The general linear group $\mathrm{Gl}(n, \mathbb{R})$ has a natural smooth manifold structure
(c) Let V, W two vector spaces and $L(V; W)$ the space of linear maps from V to W has a natural smooth manifold structure.

Exercise 2.2 (Stereographic projection.)**.** Let $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ be the *north pole* and $S = -N$ the *south pole* of the sphere $\mathbb{S}^n$. Define stereographic projection $\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x_0, \dots, x_n) = \frac{1}{1 - x_n}(x_0, \dots, x_{n-1}).$$

Let $\widetilde{\sigma}(x) = \sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(a) Show that σ is bijective, and

$$\sigma^{-1}(u_0, \dots, u_{n-1}) = \frac{1}{|u|^2 + 1}(2u_0, \dots, 2u_{n-1}, |u|^2 - 1).$$

(b) Verify that $\{\sigma, \widetilde{\sigma}\}$ is a smooth atlas for $\mathbb{S}^n$.
(c) Show that the smooth structure defined by the atlas $\{\sigma, \widetilde{\sigma}\}$ is the same as the one defined via graph coordinates in the lecture.

Exercise 2.3. Let N be an open subset of a smooth n-manifold $(M, \mathcal{A})$, endowed with the smooth structure described in Exercise 1.4. Prove that:

(a) The inclusion map $\iota : N \hookrightarrow M$ is a smooth map of manifolds.
(b) A function $f : L \to N$ from a smooth manifold l is smooth f and only if the composite $\iota \circ f$ is smooth.

Exercise 2.4 (Properties of manifolds)**.** Show that:

(a) Let $\mathcal{A}, \mathcal{A}'$ be smooth atlases on a topological manifold M. Then $\mathcal{A}$ and $\mathcal{A}'$ determine the same smooth structure on M if and only if their union is a smooth atlas.
(b) Two smooth atlases $\mathcal{A}_1, \mathcal{A}_2$ on M are equivalent iff the following holds:
For every function $f : N \to M$ (where N is a smooth manifold), the function f is smooth as a map $N \to M_0$ if and only if it is $\mathcal{C}^k$ as a map $N \to M_1$.

Exercise 2.5 (**to hand in**)**.** Prove the following

(a) Let $c : M \to N$ the constant map between two smooth manifolds; c is smooth
(b) Every smooth chart $\varphi : U \to \varphi(U)$ of M is a diffeomorphism; here U and $\varphi(U)$ are given the open subspace smooth structure defined in Exercise 1.4.

(c) The composite $g \circ f$ of two smooth maps $f : M \to N$, $g : N \to P$ is smooth map.

(d) Show that the quotient map $\pi : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ is a smooth map of manifolds where on $R\mathbb{P}^n$ we considered the smooth structure defined in Exercise 1.7.

Exercise 2.6. On the real line $\mathbb{R}$ (with the standard topology) we define two atlases $\mathcal{A} = \{\mathrm{id}_{\mathbb{R}}\}$, $\mathcal{B} = \{\varphi\}$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x) = x^3$.

(a) Find a smooth diffeomorphism $(\mathbb{R}, \overline{\mathcal{A}}) \to (\mathbb{R}, \overline{\mathcal{B}})$.