Vector bundles

→ This is a theory which has been developped to apply lots of the techniques we know from $\boxed{\text{linear algebra to manifolds}}$ linear algebra in families

→ tangent vector, differential forms, tensors, metric ... on a manifold.

§. Definition & examples

Definition: Let M a smooth mfd. A smooth vector bundle $E \xrightarrow{\pi} M$ is a smooth mfd E together with a surjective smooth map π s.t.

(1) $\forall p \in M$ E_p is a K-dimensional vector space/$\mathbb{R}$

(2) there exists local trivializations:

$\forall p \in M$, $\exists$ $p \in U \subseteq M$ s.t. $\exists$ a diffeomorphism

$$\boxed{\phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^K}$$ s.t.

- $p_U \circ \phi = \pi$

- $\Phi : E_q \to \{q\} \times \mathbb{R}^K$ is a isomorphism of real vector spaces

↳ smooth local trivialization

A vector bundle is trivial if $\exists$ a diffeomorphism:

$$\begin{tikzcd} E \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & M \times \mathbb{R}^K \arrow[dl, "p_M"] \\ & M & \end{tikzcd}$$

Recall

M is a smooth mfd of dim n

$\Rightarrow TM \xrightarrow{\pi} M$ is a vector bundle

$\bigsqcup_{p\in M}^{\shortparallel} T_pM$

1) We showed that T_pM is a vect. space of dim n

2) $\forall p\in M$, let (U,φ) a chart centred at p

$\Rightarrow$ let $x^1, \dots x^n$ coordinates on $\varphi(U)\subseteq\mathbb{R}^n$

and let us denote by

$$\frac{\partial}{\partial x^1}\Big|_q, \dots \frac{\partial}{\partial x^n}\Big|_q = D_{\varphi(q)}\varphi^{-1}\left(\frac{\partial}{\partial x^i}\right) \qquad \forall q\in U$$

$$\Rightarrow \quad \pi^{-1}(U) \xrightarrow{\Phi} U\times\mathbb{R}^n$$

$$\left(q, v=\sum_{i=1}^n v^i\frac{\partial}{\partial x^i}\Big|_q\right) \longmapsto \left(q, v^1, \dots v^n\right)$$

We used the change of coordinate formula to verify that TM is a smooth manifold

$$\pi^{-1}(U) \longrightarrow \varphi(U)\times\mathbb{R}^n$$

Moebius bundle

$/\!/ \; [0,1]\times\mathbb{R}/\sim \quad (0,y)\sim(1,-y)$

- Consider $E = \mathbb{R}^2/\sim$ where

$$(x,y)\sim(x',y') \iff \begin{array}{l} x = x'+n, \; n\in\mathbb{Z} \\ y = (-1)^n y' \end{array}$$

Let us show that this is a vector bundle on $S^1 = \mathbb{R}/\mathbb{Z}$

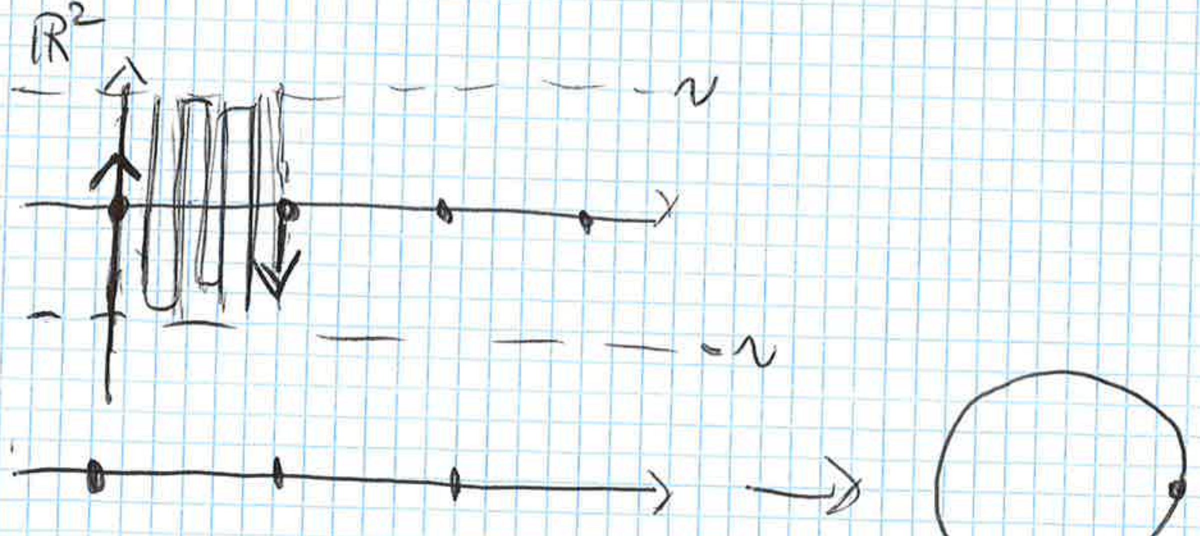
- E is sec. countable
- E is Hausdorff (check)

We have a natural map

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "q"] \arrow[d, "\pi_x"'] \arrow[dr, phantom, "?"] & E \arrow[d, "\pi"] \\ \mathbb{R}^1 \arrow[r, "e^{2\pi i x}"'] & S^1 = \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

Let us put on E a structure s.t. $E \xrightarrow{\pi} S^1$ is a smooth v.b.

- $\pi^{-1}(x) \cong \mathbb{R}$



$e^{2\pi i x}$

- check that the transition functions are smooth

Lemma (change of charts relative to ϕ)

Let $E \xrightarrow{\pi} M$ a vector bundle of rank k (smooth.)

Suppose $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, $\psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two local trivializations with $U \cap V \neq \emptyset$

$$\Rightarrow \boxed{\phi \circ \psi^{-1}(p, v) = (p, \tau(p) v)}$$ transition function.

where $\tau(p) \in GL(k, \mathbb{R})$ i.e.

$$\phi \circ \psi^{-1} \underset{\text{data of}}{\longleftrightarrow} \underline{\text{smooth map}} \; \boxed{\tau : U \cap V \to GL(k, \mathbb{R})}$$

Proof

$$\begin{tikzcd}
(U\cap V) \times \mathbb{R}^k \arrow[r, "\psi^{-1}"] \arrow[dr, "\pi_1"'] & \pi^{-1}(U \cap V) \arrow[r, "\phi"] \arrow[d, "\pi"] & (U \cap V) \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\
 & U \cap V &
\end{tikzcd}$$

$$\Rightarrow \pi_1 \circ (\phi \circ \psi^{-1}) = \pi_1$$

$$\Rightarrow \phi \circ \psi^{-1}(p, v) = (p, \sigma(p, v))$$ for

$\sigma : U \cap V \times \mathbb{R}^k \to \mathbb{R}^k$. But $\forall p$ fixed

$v \mapsto \sigma(p, v)$ is $\underline{\text{invertible linear}}$

- $\psi^{-1} \circ \phi$ has to be a diff
- $\forall p$, ϕ, ψ are vector bundle isomorphism

$$\Rightarrow \sigma(p, v) = \tau(p) \cdot v \qquad \tau(p) \in GL_n(k)$$

Example

When $E = TM$ then we saw (Lecture 4) that $\tau(p)$ is the Jacobian matrix $\left(\frac{\partial z^j}{\partial x^i}\right)_{i,j=1}^{n}$ of the change of coordinates

The next lemma will allow us to construct lots of examples of vector bundles starting from a given one

Lemma (Vector bundle chart Lemma)

Let M be a smooth manifold, $E = \bigsqcup_{p \in M} E_p \xrightarrow{\pi} M$ with E_p real vector spaces of dim K.

Suppose

(1) $\exists$ $\{U_\alpha\}_{\alpha \in A}$ cover of M s.t. $\forall \alpha$

$\exists$ $\boxed{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^K}$ is a bijection that restricts to a v.s. iso $E_p \to p \times \mathbb{R}^K$

(2) $\forall \alpha, \beta$ $U_\alpha \cap U_\beta \neq \emptyset$, $\exists$ $\boxed{\text{smooth map}}$

$$\boxed{\begin{array}{l} \tau_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(K, \mathbb{R}) \text{ t.c.} \\ \phi_\alpha \circ \phi_\beta^{-1}(p, v) = (p, \tau_{\alpha,\beta}(p)\, v) \end{array}}$$

$\Rightarrow \exists!$ smooth mfd structure on E making into a vector bundle on M with local trivialization $\{(U_\alpha, \Phi_\alpha)\}$

proof (sketch)

→ $\forall$ $p \in U_\alpha$, take $(V_p, \varphi_p) \subseteq U_\alpha$ a smooth chart for M

→ Define $\tilde{\varphi}_p : \pi^{-1}(V_p) \xrightarrow{\Phi_\alpha|_{V_p}} V_p \times \mathbb{R}^k \xrightarrow{\varphi_p \times Id_{\mathbb{R}^k}} \hat{V}_p \times \mathbb{R}^k$

→ It is not difficult to verify that $\{\pi^{-1}(V_p), \tilde{\varphi}_p\}$ satisfy the smooth chart lemma hp. (Exercise) ∎

<u>Exercise</u>: The direct sum $E \oplus E'$ of v.b is a v.b. (use lemma)

§ local & global section, framing

<u>Definition</u>: let $E \xrightarrow{\pi} M$ be a vector bundle. A section of E is a smooth map

$$\sigma : M \to E$$

s.t. $\boxed{\pi \circ \sigma = id_M}$.

A local section is a smooth map

$$\sigma : \underset{\substack{\text{open} \\ \cap \\ M}}{U} \to E \quad \text{s.t.} \quad \pi|_U \circ \sigma = id_U$$

Rmk
since By definition of vector bundle
Each Fiber E_p is a vector space
we can always have the 0-section

$$\zeta : M \to E$$
$$p \to 0 \in E_p$$

Exercise, The 0-section of a smooth vector bundle is smooth

Definition, We call a (local) section of TM a vector field

- We denote by $\mathfrak{X}(M)$ the vector space of global vector fields on M

- More general vector bundle, the vector space of global section is denoted by $\Gamma(E)$

Definition, let $E \xrightarrow{\pi} M$ a vector bundle and let $U \subseteq M$ open. A k-tuple of local section $(\sigma_1, \ldots \sigma_k) : U \to E$ is called a local frame for E over U if $\forall p \in U$ $(\sigma_1(p), \ldots \sigma_k(p))$ is a basis for E_p

IF $U = M$, then we call it <u>global frame</u>

IF TM has a global frame we say that TM is parallelizable.

<u>Example</u>

let $E = M \times \mathbb{R}^k \to M$ the trivial bundle.

$\Rightarrow \tilde{e}_i(p) := (p, e_i)$ for e_i standard coord vect. gives a global frame

<u>Proposition</u> [local frame = local trivialization]

let $E \xrightarrow{\pi} M$ be a smooth vector bundle.

(1) IF $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a local trivialization $\Rightarrow$ we can construct a local frame on U

(2) conversely, given $\sigma_1, \dots \sigma_k : U \to \pi^{-1}(U)$ a local frame

$\Rightarrow \exists\ \phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ local trivialization.

<u>proof 1</u>

(1)
$$
\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\phi"] \arrow[dr, shift left, "\pi"] & U \times \mathbb{R}^k \arrow[d, "pr_1"'] \\
& U \arrow[ul, shift left, "\sigma_i"] \arrow[u, bend right, "\tilde{e}_i"']
\end{tikzcd}
$$

Define $\boxed{\sigma_i(p) = \phi^{-1} \circ \tilde{e}_i(p)}$

- σ_i is smooth because $\tilde{e}_i$ is smooth and ϕ is a diffeomorphism by def.

→ check that

$$\pi \circ \sigma_i = id_U$$

$$\pi \circ \sigma_i (p) = \pi \circ (\phi^{-1} \circ \tilde{e}_i)(p) =$$

$$= \pi \circ (\phi^{-1}(p, e_i)) = \pi_1 (p, e_i) = p.$$

$\{\sigma_i\}$ is the local framing associated to Φ.

(2) Viceversa, let $\{\sigma_1, .. \sigma_k\}$ a local framing on U. Define

$$\psi : U \times \mathbb{R}^k \longrightarrow \pi^{-1}(U)$$

$$(p, v^1, .. v^k) \longmapsto \sum v^i \sigma_i(p)$$

⇒

(1) ψ is bijective

(2) $\sigma_i = \psi \circ \tilde{e}_i$ by commutativity of

$$\begin{tikzcd} U \times \mathbb{R}^k \arrow[r, "\psi"] \arrow[dr, "p_1"'] & \pi^{-1}(U) \arrow[d, "\pi"] \\ & U \end{tikzcd}$$

(3) we want to show ψ is a diffeomorphism $\overset{}{\underset{bij}{\Leftrightarrow}}$ local diffeomorphism

- since E is a vector bundle $\forall q \in U$, $\exists q \in V \subseteq U$ s.t.

$\exists$ a local trivialization $\Phi : \pi^{-1}(V) \to V \times \mathbb{R}^K$

$\Rightarrow \phi$ is a <u>diffeomorphism</u>

$$\begin{tikzcd} V \times \mathbb{R}^K \arrow[r, "\psi|_V"] \arrow[dr, "\pi_1"'] & \pi^{-1}(V) \arrow[r, "\phi"] \arrow[d, "\pi"] & V \times \mathbb{R}^K \arrow[dl, "\pi_1"] \\ & V & \end{tikzcd}$$

Enough to show that $\phi \circ \psi$ is a diffeomap.

- By def of section

$$\phi \circ \sigma_i = (p, \underbrace{\sigma_i^1(p), \ldots \sigma_i^K(p)}_{\text{smooth}})$$

cpts of σ_i

$$\Rightarrow \phi \circ \psi (p, v_1, \ldots v_K) = \Big(p, \underbrace{\sum_{i=1}^K v^i \sigma_i^1(p) \ldots \sum_{i=1}^K v^i \sigma_i^K(p)}_{\text{smooth}}\Big)$$

$$\Leftrightarrow \psi \circ \phi = \begin{pmatrix} \sigma_1^1 & \sigma_n^1 \\ \vdots & \\ \sigma_n^1 & \ldots \sigma_n^n \end{pmatrix}$$

$\Rightarrow (\psi \circ \phi)^{-1}$ is given by the inverse matrix which is well defined since $\{\sigma_i\}$ are a frame.

$\Rightarrow$ since inversion of matrices is a smooth map from $GL(k, \mathbb{R})$ to itself

$\Rightarrow$ ✓ ☐

Corollary

A smooth vector bundle is trivial $\Leftrightarrow$ it admits a global framing

Example

The moebious strip is not trivial

suppose it is

$\Rightarrow \exists\, \sigma : S^1 \to E$ s.t. σ is never 0

with $\sigma(x) = ([x], f(x))$ for $f : \mathbb{R} \to \mathbb{R}$ a smooth function which has to satisfy

$$\boxed{f(x+1) = -f(x)} \quad (*)$$

(to see that σ must have the form we claim it has, look in local coords)

$\Rightarrow$ If f is smooth and satisf $(*)$

$\Rightarrow$ By the intermediate valued theorem it vanishes somewhere between 0, 1

- Given $E \to M$, $E' \to M'$ vect. bundles a bundle homomorphism $F: E \to E'$ is

$$\begin{tikzcd} E \arrow[r, "F'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "F'"] & M' \end{tikzcd}$$

with F' smooth, s.t. $F'_p : E_p \to E'_{F'(p)}$ is linear.

- A bundle homomorphism over M is

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"'] & \circlearrowright & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

- Recall that given $M \xrightarrow{F} N$

$$\begin{tikzcd} DF: TM \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & TN \arrow[d] \\ F: M \arrow[r] & N \end{tikzcd}$$ Bundle homomorphism

Theorem: given $F: E \to E'$ a smooth
$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & E' \arrow[dl] \\ & M & \end{tikzcd}$$

bundle monomorphism

$\Rightarrow \ker F = \bigcup_p \ker F_p$ and $\operatorname{Im} F = \bigcup_p \operatorname{Im} F_p$
are smooth bundle over M
$\Leftrightarrow$ rank F constant.

<u>proof sketch</u>

- $\Rightarrow$ ✓
- $\Leftarrow$ It is enough to show how from local frames of E, E' we can get a local frame of $\ker F$, $\operatorname{Im} F$

$\boxtimes$