Liquid Democracy: An Algorithmic Perspective

Anson Kahng
Computer Science Department
Carnegie Mellon University

Simon Mackenzie
Computer Science Department
Carnegie Mellon University

Ariel D. Procaccia
Computer Science Department
Carnegie Mellon University

Abstract

We study liquid democracy, a collective decision making paradigm that allows voters to transitively delegate their votes, through an algorithmic lens. In our model, there are two alternatives, one correct and one incorrect, and we are interested in the probability that the majority opinion is correct. Our main question is whether there exist delegation mechanisms that are *guaranteed* to outperform direct voting, in the sense of being always at least as likely, and sometimes more likely, to make a correct decision. Even though we assume that voters can only delegate their votes to better-informed voters, we show that *local* delegation mechanisms, which only take the local neighborhood of each voter as input (and, arguably, capture the spirit of liquid democracy), cannot provide the foregoing guarantee. By contrast, we design a non-local delegation mechanism that does provably outperform direct voting under mild assumptions about voters.

1 Introduction

Liquid democracy is a modern approach to voting in which voters can either vote directly or delegate their vote to other voters. In contrast to the classic *proxy voting* paradigm (Miller 1969), the key innovation underlying liquid democracy is that proxies — who were selected by voters to vote on their behalf — may delegate their own vote to a proxy, and, in doing so, further delegate all the votes entrusted to them. Put another way (to justify the liquid metaphor), votes may freely flow through the directed delegation graph until they reach a sink, that is, a vertex with outdegree 0. When the election takes place, each voter who did not delegate his vote is weighted by the total number of votes delegated to him, including his own. In recent years, this approach has been implemented and used on a large scale, notably by eclectic political parties such as the German Pirate Party (Piratenpartei) and Sweden's Demoex (short for Democracy Experiment).

One reason for the success of liquid democracy is that it is seen as a practical compromise between *direct democracy* (voters vote directly on every issue) and *representative democracy*, and, in a sense, is the best of both worlds. Direct democracy is particularly problematic, as nicely articulated by Green-Armytage (2015):

> "Even if it were possible for every citizen to learn everything they could possibly know about every political issue, people who did this would be able to do little else, and massive amounts of time would be wasted in duplicated effort. Or, if every citizen voted but most people did not take the time to learn about the issues, the results would be highly random and/or highly sensitive to overly simplistic public relations campaigns."

By contrast, under liquid democracy, voters who did not invest an effort to learn about the issue at hand (presumably, most voters) would ideally delegate their votes to well-informed voters. This should intuitively lead to collective decisions that are less random, and more likely to be correct, than those that would be made under direct democracy.

Our goal is to rigorously investigate the intuition that liquid democracy "outperforms" direct democracy from an algorithmic viewpoint. Indeed, we are interested in *delegation mechanisms*, which decide how votes should be delegated based on how relatively informed voters are, and possibly even based on the structure of an underlying social network. Our main research question is

> *... are there delegation mechanisms that are* guaranteed *to yield more accurate decisions than direct voting?*

Overview of the Model and Results

We focus on a (common) setting where there a decision is to be made on a binary issue, i.e., one of two alternatives must be selected (see Section 5 for a discussion of the case of more than two alternatives). To model the idea of accuracy, we assume that one alternative is correct, and the other is incorrect. Each voter i has a competence level p_i, which is the probability he would vote correctly if he cast a ballot himself.

Voters may delegate their votes to neighbors in a social network, represented as a directed graph. At the heart of our model is the assumption that voters may only delegate their votes to strictly more competent neighbors (and, therefore, there can be no delegation cycles). Specifically, we say that voter i *approves* voter j if $p_j > p_i + \alpha$, for a parameter $\alpha \geq 0$; voters may only delegate to approved neighbors. In defense of this strong assumption, we note that the first of our two theorems — arguably the more interesting of the two — is an *impossibility* result, so assuming that delegation necessarily boosts accuracy only *strengthens* it.

As mentioned above, we are interested in studying *delegation mechanisms*, which decide how votes are delegated (possibly randomly), based on the underlying graph and the approval relation between voters. We pay special attention to *local* delegation mechanisms, which make delegation decisions based only on the neighborhood of each voter. Local mechanisms capture the spirit of liquid democracy in that voters make independent delegation decisions based solely on their own viewpoint, without guidance from a central authority. By contrast, non-local mechanisms intuitively require a centralized algorithm that coordinates delegations.

Recall that our goal is to design delegation mechanisms that are *guaranteed* to be more accurate than direct voting. To this end, we define the *gain* of a mechanism with respect to a given instance as the difference between the probability that it makes a correct decision (when votes are delegated and weighted majority voting is applied) and the probability that direct voting makes a correct decision on the same instance. The desired guarantee can be formalized via two properties of mechanisms: *positive gain (PG)*, which means that there are some sufficiently large instances in which the mechanism has positive gain that is bounded away from 0; and *do no harm (DNH)*, which requires that the loss (negative gain) of the mechanism goes to 0 as the number of voters grows. These properties are both weak; in particular, PG is a truly minimal requirement which, in a sense, mainly rules out direct voting itself as a delegation mechanism.

In Section 3, we study local delegation mechanisms and establish an impossibility result: such mechanisms cannot satisfy both PG and DNH. In a nutshell, the idea is that for any local delegation mechanism that satisfies PG we can construct an instance where few voters amass a large number of delegated votes, that is, delegation introduces significant *correlation* between the votes. The instance is such that, when the high-weight voters are incorrect, the weighted majority vote is incorrect; yet direct voting is very likely to lead to a correct decision.

In Section 4, we show that non-local mechanisms can circumvent the foregoing impossibility. Specifically, we design a delegation mechanism, GREEDYCAP, that satisfies the PG and DNH properties under mild assumptions about voter competencies. It does so by imposing a cap on the number of votes that can be delegated to any particular voter, thereby avoiding excessive correlation.

In conclusion, our work highlights the significance, and potential dangers, of delegating many votes to few voters. Importantly, there is evidence that this can happen in practice. For example, Der Spiegel reported[1] that one member of the German Pirate Party, a linguistics professor at the University of Bamberg, amassed so much weight that his "vote was like a decree." Our results corroborate the intuition that this situation should be avoided.

Related Work

There is a significant body of work on delegative democracy and proxy voting (Miller 1969; Tullock 1992; Alger 2006; Cohensius et al. 2017). But, to the best of our knowledge, there are only two papers that provide theoretical analyses of liquid democracy. The first is the aforementioned paper by Green-Armytage (2015). He considers a setting where voters' positions on an issue are represented as points on the real line and votes are noisy estimates of those positions. Green-Armytage defines the *expressive loss* of a voter as the squared distance between his vote and his position and proves that delegation (even transitive delegation) can only decrease the expressive loss in his model. He also defines *systematic loss* as the squared distance between the median vote and the median position, but discusses this type of loss only informally (interestingly, he does explicitly mention that correlation can lead to systematic loss in his model).

The second paper is by Christoff and Grossi (2017). They introduce a model of liquid democracy based on the theory of binary aggregation (i.e., their model has a mathematical logic flavor). Their results focus on two problems: the possibility of delegation cycles, and logical inconsistencies that can arise when opinions on interdependent propositions are expressed through proxies. Both are nonissues in our model (although the need to avoid cycles is certainly a concern in practice).

Further afield, there is a rich body of work in computational social choice (Brandt et al. 2016) on the aggregation of objective opinions (Conitzer and Sandholm 2005; Conitzer, Rognlie, and Xia 2009; Elkind, Faliszewski, and Slinko 2010; Elkind and Shah 2014; Xia, Conitzer, and Lang 2010; Xia and Conitzer 2011; Lu and Boutilier 2011; Procaccia, Reddi, and Shah 2012; Azari Soufiani, Parkes, and Xia 2012; 2013; 2014; Mao, Procaccia, and Chen 2013; Caragiannis, Procaccia, and Shah 2014; 2016; Procaccia, Shah, and Zick 2016; Xia 2016). As in our work, the high-level goal is to pinpoint the correct outcome based on noisy votes. However, previous work in this area does not encompass any notion of vote delegation.

2 The Model

We represent an instance of our problem using a directed, labeled graph $G = (V, E, \vec{p})$. $V = \{1, \ldots, n\}$ is a set of n *voters*, also referred to as *vertices* (we use the two terms interchangeably). E represents a (directed) social network in which the existence of an edge (i, j) means that voter i knows (of) voter j.

We assume that the voters vote on a binary issue; there is a correct alternative and an incorrect alternative. Each voter $i \in V$ is labeled by his *competence level* p_i. This is the

[1] http://www.spiegel.de/international/germany/liquid-democracy-web-platform-makes-professor-most-powerful-pirate-a-818683.html

probability that i has the correct opinion about the issue at hand, i.e., the probability that i will vote correctly.

Our setting is also parameterized by $\alpha \in [0, 1)$. Given this parameter and a labeled graph $G = (V, E, \vec{p})$, we define an approval relation between voters: $i \in V$ *approves* $j \in V$ if $(i, j) \in E$ and $p_j > p_i + \alpha$. In words, i approves his neighbor j if the difference in their competence levels is strictly greater than α. The strict inequality guarantees that the approval relation is acyclic. Denote

$$A_G(i) = \{j \in V : i \text{ approves } j\}.$$

Delegation Mechanisms

The liquid democracy paradigm is implemented through a *delegation mechanism* M, which takes as input a labeled graph G, and outputs, for each voter i, a delegation probability distribution over $A_G(i) \cup \{i\}$ that represents the probability that i will delegate his vote to each of his approved neighbors, or to himself (which means he does not delegate his vote).

To determine whether a delegation mechanism M makes a correct decision on a labeled graph $G = (V, E, \vec{p})$, we use the following 4-step process (which is described in words to avoid introducing notation that will not be used again):

1. Apply M to G.
2. Sample the probability distribution for each vertex to obtain an acyclic *delegation graph*. Each sink i of the delegation graph (i.e., vertex with no outgoing edges) has weight equal to the number of vertices with directed paths to i, including i itself.
3. Each sink i votes for the correct alternative with probability p_i, and for the incorrect alternative with probability $1 - p_i$.
4. A decision is made based on the weighted majority vote.[2]

We denote the probability that the mechanism M makes a correct decision on graph G via this 4-step process by $P_M(G)$.

We are especially interested in a special class of delegation mechanisms that we call *local* mechanisms. Intuitively, local mechanisms capture the natural setting where each voter makes an independent delegation decision without central coordination. Formally, a *local delegation mechanism* is a delegation mechanism such that the probability distribution of each vertex i depends only on the neighborhood of i in G, i.e., on $\{j \in V : (i, j) \in E\}$, and on $A_G(i)$, i.e., the subset of these neighbors that are approved. For example, the following delegation mechanisms are local:

- Voters do not delegate their votes. This *direct voting* mechanism plays a special role in our model, and we denote it by D.
- Each voter delegates his vote to a random approved neighbor, if he has any.
- Each voter delegates his vote to a random approved neighbor, if he has approved neighbors but has even more non-approved neighbors.
- Each voter delegates his vote deterministically to a single approved neighbor, if he has any.[3]

Desiderata

Recall that we are interested in comparing the likelihood of making correct decisions via delegative voting with that of direct voting. To this end, define the *gain* of delegation mechanism M on labeled graph G as

$$\text{gain}(M, G) = P_M(G) - P_D(G).$$

We would like to design delegation mechanisms that have positive gain (bounded away from zero) in some situations, and which never lose significantly to direct voting. Formally, we are interested in the following two desirable axioms:

- A mechanism M satisfies the *positive gain (PG)* property if there exist $\gamma > 0, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there exists a graph G_n on n vertices such that $\text{gain}(M, G_n) \geq \gamma$.
- A mechanism M satisfies the *do no harm (DNH)* property if for all $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that for all graphs G_n on $n \geq n_1$ vertices, $\text{gain}(M, G_n) \geq -\varepsilon$.

The choice of quantifiers here is of great significance. PG asks for the *existence* of (large enough) instances where the gain is at least γ, for a constant γ. By contrast, DNH essentially requires that any loss would go to 0 as the size of the graph goes to infinity. That is, there may certainly be small instances where delegative voting loses out to direct voting, but that should not be the case in the large.

3 Impossibility for Local Mechanisms

In our model, we make the strong assumption that voters can only delegate their vote to other voters who are more competent than they are, and, in particular, delegation chains can significantly boost the competence of any particular vote. Under this assumption, it seems natural to expect that delegative voting will always do at least as well as direct voting in every situation, and strictly better in some situations. This should intuitively be true under local mechanisms, say, when each voter delegates his vote to an arbitrary approved neighbor (if he has any). The following example helps build intuition for what can go wrong.

[2] Ties can be broken arbitrarily.

[3] There is a technical subtlety here: To implement such a local mechanism, vertices cannot be anonymous, so we require an *ordering* over the approved neighbors of each vertex, e.g., the one induced by the indices. We do not belabor this point but note that it is not an issue for our technical results.

Example 1. Consider the labeled graph $G_n = (V, E, \vec{p})$ over n vertices, where $E = \{(i, 1) : i \in V \setminus \{1\}\}$, i.e., G is a star with 1 at the center. Moreover, $p_1 = 4/5$, $p_i = 2/3$ for all $i \in V \setminus \{1\}$, and $\alpha = 1/10$. Then, as n grows larger, $P_D(G_n)$ goes to 1 by the Law of Large Numbers, or, equivalently, by the Condorcet Jury Theorem (Grofman, Owen, and Feld 1983). By contrast, all leaves approve the center, and a naïve local delegation mechanism M would delegate all their votes. In that case, the decision depends only on the vote of the center, so $P_M(G_n) = 4/5$ for all $n \in \mathbb{N}$, and gain(M, G_n) converges to $-1/5$. We conclude that M violates the DNH property.

One might hope that there are "smarter" local delegation mechanisms, that, say, recognize that when a voter only has one approved neighbor, his vote should not be delegated. However, our first result shows that this is not the case: local delegation mechanisms cannot even satisfy the two minimal requirements of PG and DNH.

Theorem 1. *For any $\alpha_0 \in [0, 1)$, there is no local mechanism that satisfies the PG and DNH properties.*

The first step in the proof is better understanding the way in which local mechanisms are constrained. This is captured by the following lemma.

Lemma 1. *Let M be a local mechanism. Then M satisfies the PG property only if there exist $k, m, \rho > 0$ such that, if a voter approves of exactly k of his m neighbors, then the total probability of delegation to any of these approved neighbors is exactly ρ.*

Proof. Suppose that PG holds. Let $\gamma > 0$ and fix a labeled graph G such that gain$(M, G) \geq \gamma > 0$. In order for this to be the case, there must exist some vertex i that delegates with positive probability ρ. Let k be the number of neighbors in G that i approves, and let m be his total number of neighbors in G; this yields the desired tuple (k, m, ρ). □

The crux of the theorem's proof is the construction of a graph that, from the local viewpoint of many of the vertices, looks like the neighborhood prescribed by Lemma 1. Specifically, a *k-center m-uniform star* consists of vertices called *leaves* that are each connected to k central vertices (the centers) as well as $m - k$ other leaves. Each leaf vertex has competence level p_ℓ, and each center vertex has competence level p_c, such that $p_c > p_\ell + \alpha$. We set the value of k and m to be the values whose existence is guaranteed by Lemma 1, which means that the construction of a k-center m-uniform star satisfies the property that each leaf delegates to some center vertex with probability ρ. Throughout the proof, we will let $n_c = k$ be the number of centers, and n_ℓ will denote the number of leaves.

At a high level, we show that the loss of any local mechanism can approach $(1 - p_c)^k$, which is constant given k. We do this by constructing a graph that consists of a k-center m-uniform star with an independent disconnected component consisting of n_d vertices of competence level p_d. We set the parameters so that the direct voting mechanism D decides correctly with high probability. By contrast, under the local delegation mechanism M, enough leaves delegate their votes to the centers so that if all centers were to vote incorrectly, which happens with probability $(1 - p_c)^k$, then M would decide incorrectly. While the basic idea is simple enough, the formal construction is quite delicate, as many different parameters must be perfectly balanced.

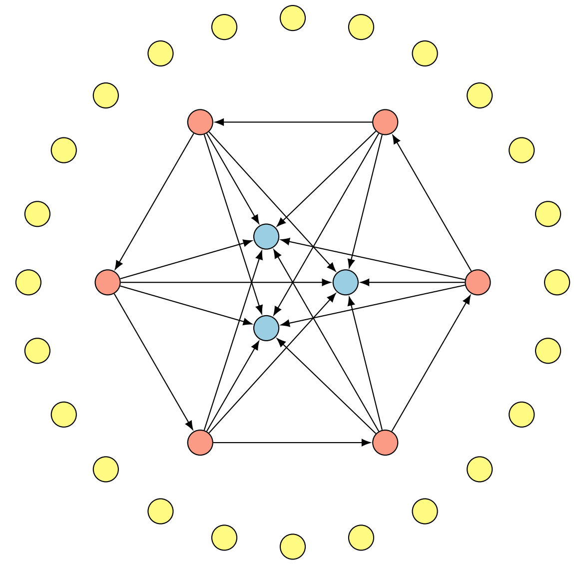

Figure 1: Graph G for $n_\ell = 6$ leaves (shown in red), $n_c = 3$ centers (shown in blue), $n_d = 24$ disconnected vertices (shown in yellow), and $m = 4$.

Proof of Theorem 1. Let M be a local mechanism that satisfies PG. By Lemma 1, there must exist at least one (k, m, ρ) tuple for M that satisfies the lemma's conclusion. For any n_1 prescribed by DNH and any α_0, we can construct a graph G_n such that DNH does not hold.

Let G be a graph of size $n = n_c + n_\ell + n_d$ that consists of a k-center m-uniform star and a disconnected component containing n_d disconnected points (see Figure 1). Each center has competence level p_c, each leaf in the star has competence level p_ℓ, and each point in the disconnected component has competence level p_d. Given (k, m, ρ), n_1, and α_0, note that the following constraints must hold.

$$n_\ell \geq m - n_c \tag{1}$$

$$n = n_\ell + n_c + n_d \geq n_1 \tag{2}$$

$$p_c > p_\ell + \alpha_0 \tag{3}$$

We will prove that the following explicit construction violates DNH for any input of (k, m, ρ), n_1, and $\alpha = \alpha_0 + \varepsilon'$ for $\varepsilon' = \frac{1-\alpha_0}{2} > 0$, as $\delta \to 0$.

$$n_c = k \tag{4}$$

$$n_\ell = \frac{n_1 m}{\alpha\delta} \tag{5}$$
$$n_d = C_1 \frac{n_1 m}{\alpha\delta} \tag{6}$$
$$C_1 = \frac{\left(\frac{\left(\frac{p_\ell \rho}{\sigma} n_\ell - p_\ell \sqrt{n_\ell}\right)}{2}\right)^2 - n_c}{n_\ell} - 1 \tag{7}$$
$$\sigma = \sqrt{-\frac{\ln\left(\frac{\delta}{2}\right)}{2}} \tag{8}$$
$$p_c = \frac{1+\alpha}{2} \tag{9}$$
$$p_l = \frac{1-\alpha}{2} \tag{10}$$
$$p_d \in \left[\left(\frac{n/2 - n_\ell p_\ell}{n_d}\right) + \frac{\sigma\sqrt{n}}{n_d}, \right. \tag{11}$$
$$\left. \left(\frac{n/2 - n_\ell p_\ell}{n_d}\right) + \frac{(n_\ell\rho - \tau)p_\ell - \sigma\sqrt{n}}{n_d}\right]$$

Note that we define τ in (12) below. The following claim, whose proof is relegated to Appendix A, asserts that the construction is feasible.

Claim 1. *$C_1 > 0$ and the range of values for p_d in* (11) *is nonempty.*

Because $\alpha, \delta \in (0, 1)$ and $C_1 > 0$, the value of n_ℓ in (5) is greater than both n_1 and m, hence constraints (1) and (2) are immediately satisfied. Moreover, constraint (3) is satisfied by (9) and (10).

Turning to the proof that DNH is violated, let S, Z, and W be the random variables corresponding to the number of correct votes under D, the number of delegated votes under M, and the number of non-delegated correct votes under M. Additionally, let ε, τ, and ξ be as follows.

$$\begin{aligned} \varepsilon &= \sqrt{-\frac{\left(\ln\frac{\delta}{2}\right)n}{2}}, \\ \tau &= \sqrt{-\frac{\left(\ln\frac{\delta}{2}\right)n_\ell}{2}}, \text{ and} \\ \xi &= \sqrt{-\frac{\left(\ln\frac{\delta}{2}\right)(n - n_c - (\rho n_\ell - \tau))}{2}}. \end{aligned} \tag{12}$$

Our goal is to bound the expectations of S, Z, and W. First, we examine $\mathbb{E}[S]$. We would like to show that

$$\mathbb{E}[S] \geq n/2 + \varepsilon. \tag{13}$$

Expanding out the expected value, this is equivalent to

$$p_c n_c + p_\ell n_\ell + p_d n_d \geq n/2 + \varepsilon.$$

From (11), we have

$$p_d \geq \frac{n/2 - p_\ell n_\ell + \varepsilon}{n_d},$$

so it is sufficient to show that

$$p_c n_c + p_\ell n_\ell + n_d\left(\frac{n/2 - p_\ell n_\ell + \varepsilon}{n_d}\right) \geq n/2 + \varepsilon,$$

and simplifying results in $p_c n_c \geq 0$. This is true by Equation (9), because α and k are both constrained to be strictly positive.

Next, we examine $\mathbb{E}[Z]$. We would like to show that

$$\mathbb{E}[Z] = n_\ell \rho. \tag{14}$$

This is trivial to see, as Z is a sum of n_ℓ Bernoulli random variables with "success" probability ρ.

Finally, we examine the "typical case" over W, or $\mathbb{E}[W|Z = v]$ for all integers $v \in [n_\ell\rho - \tau, n_\ell\rho + \tau]$. Intuitively, this is examining the number of correct votes cast by still-independent vertices after "enough" leaf vertices have delegated their votes. If these votes do not make up a majority, then all centers voting incorrectly will cause the entire graph to vote incorrectly. We would like to show that

$$\mathbb{E}[W|Z = v] \leq n/2 - \xi. \tag{15}$$

for all integers $v \in [n_\ell\rho - \tau, n_\ell\rho + \tau]$. Conditionally on Z being in the prescribed range above, we see that in the worst case, $Z = n_\ell\rho - \tau$, meaning the fewest possible voters delegate under this assumption. Given this, we would like to show that

$$p_d n_d + p_\ell(n_\ell - (\rho n_\ell - \tau)) \leq n/2 - \xi.$$

From Equation (11) we have

$$p_d \leq \frac{n/2 - p_\ell n_\ell + (n_\ell\rho - \tau)p_\ell - \xi}{n_d},$$

which yields

$$\left(\frac{n/2 - p_\ell n_\ell + (n_\ell\rho - \tau)p_\ell - \xi}{n_d}\right) n_d + p_\ell(n_\ell - (\rho n_\ell - \tau)) \leq n/2 - \xi.$$

Simplifying results in $0 \leq 0$ — a tautology. This establishes Equation (15).

We now wish to bound the probability of S, Z, and W deviating by too much. We use Hoeffding's inequality, which states that given n independent Bernoulli random variables $X_i \in [0, 1]$ and $X = \sum_i X_i$, the following concentration bound holds:

$$\Pr\left[|X - \mathbb{E}[X]| \geq \varepsilon\right] \leq 2\exp\left(\frac{-2\varepsilon^2}{n}\right). \tag{16}$$

First, we examine S. From (16) and a straightforward substitution for ε, we obtain

$$\begin{aligned} \Pr\left(|S - \mathbb{E}[S]| \geq \varepsilon\right) &\leq 2\exp\left(\frac{-2\varepsilon^2}{n}\right) \\ &= 2\exp\left(-\frac{2\left[\sqrt{-\frac{\left(\ln\frac{\delta}{2}\right)n}{2}}\right]^2}{n}\right) \\ &= \delta. \end{aligned} \tag{17}$$

Likewise, for Z, from (16) and a straightforward substitution for τ, we obtain

$$\begin{aligned}\Pr\left[|Z-\mathbb{E}[Z]|\geq\tau\right] &\leq 2\exp\left(\frac{-2\tau^2}{n_\ell}\right)\\ &= 2\exp\left(-\frac{2\left[\sqrt{-\frac{\left(\ln\frac{\delta}{2}\right)n_\ell}{2}}\right]^2}{n_\ell}\right)\\ &= \delta.\end{aligned} \tag{18}$$

Finally, for W, we are interested in upper-bounding

$$\Pr[|W-\mathbb{E}[W|Z=v]|\geq\xi \mid Z=v],$$

for every integer $v\in[n_\ell\rho-\tau, n_\ell\rho+\tau]$. As before, we apply Equation (16), and, as it turns out, we can derive an upper bound when $Z=n_\ell\rho-\tau$. Therefore, we obtain that for every $v\in[n_\ell\rho-\tau, n_\ell\rho+\tau]$,

$$\begin{aligned}&\Pr\left[|W-\mathbb{E}[W|Z=v]|\geq\xi \mid Z=v\right]\\ &\quad\leq 2\exp\left(\frac{-2\xi^2}{n-n_c-(\rho n_\ell-\tau)}\right)\\ &\quad= 2\exp\left(-\frac{2\left[\sqrt{-\frac{\left(\ln\frac{\delta}{2}\right)(n-n_c-(\rho n_\ell-\tau))}{2}}\right]^2}{n-n_c-(\rho n_\ell-\tau)}\right)\\ &\quad= \delta.\end{aligned} \tag{19}$$

From the above, we see that

$$\begin{aligned}\Pr[S>n/2] &\geq 1-\delta, \quad \text{(by (13) and (17))}\\ \Pr[Z\in(n_\ell\rho-\tau, n_\ell\rho+\tau)] &\geq 1-\delta, \quad \text{(by (14) and (18))}\\ \Pr[W<n/2 \mid Z=v] &\geq 1-\delta, \quad \text{(by (15) and (19))}\end{aligned}$$

where the last inequality holds for all integers $v\in[n_\ell\rho-\tau, n_\ell\rho+\tau]$.

Therefore, the lower bound on the probability of D deciding correctly is $P_D(G)\geq 1-\delta$. We can lower-bound the probability of M deciding incorrectly in order to upper-bound $P_M(G)$. We slightly overload notation and let M be the event that M decides correctly, and $\neg M$ be the event that M decides incorrectly. Moreover, denote by V the event that $Z\in[n_\ell\rho-\tau, n_\ell\rho+\tau]$. By definition, we have

$$\Pr[\neg M]=\Pr[\neg M|V]\Pr[V]+\Pr[\neg M|\neg V]\Pr[\neg V],$$

and because probabilities cannot be negative,

$$\Pr[\neg M]\geq\Pr[\neg M|V]\Pr[V].$$

Now, because $\Pr[V]\geq 1-\delta$,

$$\Pr[\neg M]\geq\Pr[\neg M|V](1-\delta).$$

Furthermore, we know that $\Pr[\neg M|V]$ is also lower-bounded by $(1-p_c)^{n_c}(1-\delta)$ because one setting under which M decides incorrectly is exactly when all centers vote incorrectly and $W<n/2$. It follows that

$$\Pr[\neg M]\geq(1-p_c)^{n_c}(1-\delta)(1-\delta).$$

Therefore, taking the complement, we have an upper bound on the probability of M voting correctly of

$$\Pr[M]\leq 1-(1-p_c)^{n_c}(1-\delta)^2,$$

and the total loss can be lower-bounded by

$$(1-\delta)-(1-(1-p_c)^{n_c}(1-\delta)^2)=(1-p_c)^{n_c}(1-\delta)^2-\delta.$$

As $\delta\to 0$, this tends to $(1-p_c)^{n_c}=(1-p_c)^k$, which is constant and bounded away from 0. We conclude that M violates the DNH property. $\square$

4 Possibility for Non-Local Mechanisms

The main idea underlying Theorem 1 is that liquid democracy can correlate the votes to the point where the mistakes of a few popular voters tip the scales in the wrong direction. As we show in the theorem's proof, this is unavoidable under local delegation mechanisms, which, intuitively, cannot identify situations in which certain voters amass a large number of votes. However, non-local delegation mechanisms can circumvent this issue. Indeed, consider the following delegation mechanism.

input: labeled graph G, cap $C:\mathbb{N}\to\mathbb{N}$

```
V' ← V
while V' ≠ ∅ do
    let i ∈ argmax_{j∈V'} |A_G(j) ∩ V'|
    J ← A_G(i) ∩ V'
    if |J| ≤ C(n) − 1 then
        J' ← J
    else
        let J' ⊆ J such that |J'| = C(n) − 1
    end if
    vertices in J' delegate to i
    V' ← V' \ ({i} ∪ {J'})
end while
```

Algorithm 1: GREEDYCAP

In words, the mechanism GREEDYCAP, given as Algorithm 1, receives as input a labeled graph G, and a *cap* C. It iteratively selects a voter with maximum approvals, and delegates votes to him, so that no more than $C(n)-1$ votes are delegated to a single voter (that is, no voter can have weight more than $C(n)$). All voters involved in the current iteration are then eliminated from further consideration, which is why delegations under this mechanism are only 1-hop.

It is obvious that GREEDYCAP satisfies the PG property. However, although it seems at first glance that it should satisfy DNH as well (as it solves the excessive correlation problem), the following example shows that, without further assumptions, it does not.

Example 2. Assume for ease of exposition that $\alpha < 1/3$. For any odd $n = 2k+1$, consider the labeled graph $G_n = (V, E, \vec{p})$ on n vertices, defined as follows: $E = \{(1,2)\}$ (i.e., the only edge in the graph is from 1 to 2), $p_1 = 1/3$, $p_2 = 2/3$, there are k vertices with $p_i = 1$, and $k-1$ vertices with $p_i = 0$. Even if $C(n) \equiv 2$, GREEDYCAP would delegate the vote of voter 1 to 2. Therefore, the mechanism decides correctly if and only if 2 votes correctly, which happens with probability $2/3$. By contrast, under direct voting, it is enough for either 1 or 2 to vote correctly, which happens with probability $7/9$. It follows that the loss of GREEDYCAP is $1/9$ — a constant. We conclude that GREEDYCAP violates DNH.

The reason the example works is that the outcome completely depends on voters 1 and 2, as the others vote deterministically (competence level 0 or 1). To avoid this problem, we make the natural assumption that competence levels are bounded away from 0 and 1, i.e., voters are never horribly misinformed or perfectly informed. It turns out that this additional assumption is sufficient to guarantee that GREEDYCAP satisfies the DNH property.

Theorem 2. *Assume that there exists $\beta \in (0, 1/2)$ such that all competence levels are in $[\beta, 1-\beta]$. Then for any $\alpha \in (0, 1-2\beta)$,* GREEDYCAP *with cap $C : \mathbb{N} \to \mathbb{N}$ such that $C(n) \in \omega(1)$ and $C(n) \in o(\sqrt{\log n})$ satisfies the PG and DNH properties.*

The theorem's rather technical proof is relegated to Appendix B. Here we provide a proof sketch.

The PG property is rather obvious. It suffices to construct a family of examples over which the property is satisfied. Let the underlying graph G consist of pairs of nodes with one competent voter with competence level $1-\beta$, and one incompetent voter with competence level β, where there is an edge from every incompetent voter to the connected competent voter. Under the direct setting, it is clear that $P_D = 1/2$. However, under GREEDYCAP, all incompetent voters delegate to competent voters, resulting in $n/2$ independent voters who each have competence $1-\beta$ and weight exactly two. By the Condorcet Jury Theorem (Grofman, Owen, and Feld 1983), the probability of success approaches 1.

For the DNH property, we denote the number of correct votes under direct voting and GREEDYCAP by X_D and X_M, respectively, and consider two cases.

1. $|\mathbb{E}[X_D] - \frac{n}{2}| > \frac{n}{\log n}$.
2. $|\mathbb{E}[X_D] - \frac{n}{2}| \leq \frac{n}{\log n}$.

In Case 1, the direct voting mechanism has mean far away from $n/2$. When $\mathbb{E}[X_D] < n/2 - n/\log n$, we can show that P_D goes to 0 as n goes to infinity. This means that DNH is satisfied for any value of P_M. In the case where $\mathbb{E}[X_D] > n/2 + n/\log n$, we can show that P_M goes to 1 as n goes to infinity, which means that DNH is satisfied for any value of P_D.

In Case 2, the direct voting setting has mean close to $n/2$. From here, we consider two subcases.

1. The number of voters who delegate is greater than $n/g(n)$, where $g(n) \in o(\log n)$ and $g(n) \in \omega(C(n)^2)$ — hence the upper bound on $C(n)$.
2. The number of voters who delegate is at most $n/g(n)$.

In Subcase 1, because a relatively large fraction of voters delegate their votes to more competent neighbors, $\mathbb{E}[X_M] - \mathbb{E}[X_D]$ is large enough to offset the simultaneous increase in the variance of X_M, and, in the limit, P_M goes to 1. In Subcase 2, we again have $\mathbb{E}[X_M] \geq \mathbb{E}[X_D]$ due to delegation. Additionally, because so few voters delegate, the ratio of the variance of X_M and that of X_D converges to 1 as n approaches infinity, which means that (in the worst case) the difference between P_D and P_M converges to 0.

5 Discussion

We wrap up with a discussion of two central issues.

How realistic is the model? We revisit an important point, which has already come up several times, including in Section 1. Our assumption that voters only delegate their votes to more competent voters is clearly restrictive. But we feel it allows us, in a sense, to distill the essence of liquid democracy (e.g., by avoiding complications that have to do with delegation cycles) and focus on central issues such as vote correlation. Moreover, as noted earlier, our negative result — Theorem 1 — is especially powerful in this model, that is, it holds *despite* the foregoing assumption. And the positive result — Theorem 2 — should (informally speaking) still hold in a relaxed model where voters may delegate their votes to less competent voters, as long as the average competence level increases by a constant due to delegation. We view this as a realistic assumption.

Beyond binary issues. In our model, there are only two alternatives, one correct and one incorrect. While this setting is of practical importance, it is natural to ask whether our results extend to the case of three or more alternatives. However, there are several obstacles.

First, a representation of the ground truth, and of voters' perceptions thereof, is required. A popular option is the *Mallows* (1957) *Model*, where the ground truth is a ranking of the alternatives, and the probability that a voter cast a given ranking as his vote decreases exponentially with its "distance" from the ground truth, in a way that depends on a (competence) parameter ϕ_i. This model coincides with ours (using a suitable transformation between ϕ_i and p_i) when the number of alternatives is 2.

Second, we have assumed that votes are aggregated using the majority rule, which is the only reasonable voting rule when there are two alternatives. By contrast, when choosing among three or more alternatives, there are many voting rules one can use.

To conclude, any attempt to extend our model and results beyond the case of two alternatives would have to address not only technical challenges, but also conceptual ones.

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Appendix

A Proof of Claim 1

We show that our value of C_1 results in the existence of p_d from above by relaxing the upper and lower bounds on p_d.

From above, we have

$$C_1 = \frac{\left(\frac{\left(\frac{p_\ell \rho}{\sigma} n_\ell - p_\ell \sqrt{n_\ell}\right)}{2}\right)^2 - k}{n_\ell} - 1$$

and rearranging terms yields

$$2\sqrt{(C_1+1)n_\ell + k} = \frac{p_\ell \rho}{\sigma} n_\ell - p_\ell \sqrt{n_\ell}.$$

Now, note that $n_d = C_1 n_\ell$ and therefore $(C_1+1)n_\ell + k = n_d + n_\ell + k = n$. Additionally, note that $\sqrt{n_\ell} = \frac{\tau}{\sigma}$. Substituting this in, we have

$$2\sqrt{n} = \frac{p_\ell \rho n_\ell - p_\ell \tau}{\sigma}$$

and therefore

$$\sigma\sqrt{n} = p_\ell(\rho n_\ell - \tau) - \sigma\sqrt{n}. \tag{20}$$

Now, we note that $\sigma\sqrt{n} - kp_c < \sigma\sqrt{n}$ and

$$p_\ell(\rho n_\ell - \tau) - \sigma\sqrt{n} < (n_\ell\rho - \tau)p_\ell - \sigma\sqrt{n - k - (n_\ell\rho - \tau)},$$

so from (20), we can now conclude that

$$\begin{aligned}\sigma\sqrt{n} - kp_c &< \sigma\sqrt{n} = p_\ell(\rho n_\ell - \tau) - \sigma\sqrt{n} \\ &< (n_\ell\rho - \tau)p_\ell - \sigma\sqrt{n - k - (n_\ell\rho - \tau)},\end{aligned}$$

which means

$$\begin{aligned}p_d \in \Bigg[&\left(\frac{n/2 - n_\ell p_\ell}{c}\right) + \frac{\sigma\sqrt{n} - kp_c}{c}, \\ &\left(\frac{n/2 - n_\ell p_\ell}{c}\right) + \frac{(n_\ell\rho - \tau)p_\ell - \sigma\sqrt{n - k - (n_\ell\rho - \tau)}}{c}\Bigg]\end{aligned}$$

has a solution, as desired, and our value for p_d is admissible.

Lastly, we have to show that this value of C_1 is itself admissible; i.e., that the following holds:

$$\frac{\left(\frac{\left(\frac{p_\ell\rho}{\sigma}n_\ell - p_\ell\sqrt{n_\ell}\right)}{2}\right)^2 - k}{n_\ell} - 1 > 0.$$

Rearranging and expanding, we obtain

$$\frac{p_\ell n_\ell \rho}{\sigma} - p_\ell\sqrt{n_\ell} \geq 2\sqrt{n_\ell + k},$$

and squaring both sides yields

$$\left(\frac{p_\ell n_\ell \rho}{\sigma}\right)^2 + (p_\ell)^2 n_\ell - 2\frac{(p_\ell)^2\rho(n_\ell)^{3/2}}{\sigma} \geq 4(n_\ell + k).$$

Now, substituting in our value for n_ℓ, we obtain

$$\begin{aligned}&\left[\left(\frac{p_\ell\rho}{\sigma}\right)\left(\frac{n_0 m}{\alpha\delta}\right)\right]^2 + (p_\ell)^2\left(\frac{n_0 m}{\alpha\delta}\right) - \frac{2(p_\ell)^2\rho}{\sigma}\left(\frac{n_0 m}{\alpha\delta}\right)^{3/2} \\ &\quad - 4\left(\frac{n_0 m}{\alpha\delta}\right) - 4k \\ &\geq 0.\end{aligned}$$

As $\delta \to 0$, this becomes dominated by the highest-order $1/\delta$ term, and therefore is always positive for any assignment to the other variables because the rest of them are constrained to be strictly positive. ∎

B Proof of Theorem 2

We use the following classic result.

Lemma 2 (Lyapunov)**.** *Let $\{X_1, \ldots, X_n\}$ be a set of independent random variables, each with finite expected value $\mathbb{E}[X_i]$ and variance $\operatorname{Var}[X_i]$. Let $s_n^2 = \sum_{i=1}^n \operatorname{Var}[X_i]$. If, for some $\delta > 0$ the following condition holds, then the sum $\sum_{i=1}^n \frac{X_i - \mathbb{E}[X_i]}{s_n}$ converges to a standard normal random variable as n goes to ∞:*

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^n \mathbb{E}\left[|X_i - \mathbb{E}[X_i]|^{2+\delta}\right] = 0.$$

The next lemma adapts the previous one for our setting.

Lemma 3. *Let S be a random variable such that $S = X_1 + \cdots + X_t$, where each $X_i = w_i V_i$ for $w_i \in \mathbb{Z}^+$, and each V_i is an independent Bernoulli random variable with success probability $p_i \in [\beta, 1-\beta]$ for $\beta \in (0, 1/2)$. Furthermore, $\sum_{i=1}^t w_i = n$, $n/C(n) \leq t \leq n$, and each $w_i \leq C(n)$, where $C(n) \in o(n^{1/3})$. Then $\mathbb{E}[S]$ converges to a normal distribution.*

Proof. To use Lemma 2, let $\delta = 1$; we would like to show that

$$\lim_{t\to\infty} \frac{\sum_{i=1}^t \mathbb{E}\left[|X_i - \mathbb{E}[X_i]|^3\right]}{s_t^3} = 0.$$

From above, we know that $s_t^3 = \left(\sum_{i=1}^t \operatorname{Var}[X_i]\right)^{3/2}$ and $\operatorname{Var}[X_i] = w_i^2 \operatorname{Var}[V_i] = w_i^2 p(1-p)$. Plugging this in yields

$$\lim_{t\to\infty} \frac{\sum_{i=1}^t \mathbb{E}\left[|X_i - \mathbb{E}[X_i]|^3\right]}{\left(\sum_{i=1}^t w_i^2 p_i(1-p_i)\right)^{3/2}}.$$

Additionally,

$$\mathbb{E}\left[|X_i - \mathbb{E}[X_i]|^3\right] = w_i^3\left(p_i(1-p_i)^3 + (1-p_i)(|-p_i|)^3\right),$$

which simplifies to $w_i^3 p_i(1-p_i)(1-2p_i+2p_i^2)$. Therefore, we have

$$\lim_{t\to\infty} \frac{\sum_{i=1}^t w_i^3 p_i(1-p_i)(1-2(p_i-p_i^2))}{\left(\sum_{i=1}^t w_i^2 p_i(1-p_i)\right)^{3/2}}$$

and because $p_i \in [\beta, 1-\beta]$ with $\beta \in (0, 1/2)$, we know that $p_i > p_i^2$ and so $1 - 2(p_i - p_i^2) < 1$. Therefore, for all valid p_i, $p_i(1-p_i)(1-2(p_i-p_i^2)) < p_i(1-p_i)$, and we can see that

$$\lim_{t\to\infty} \frac{\sum_{i=1}^t w_i^3 p_i(1-p_i)(1-2(p_i-p_i^2))}{\left(\sum_{i=1}^t w_i^2 p_i(1-p_i)\right)^{3/2}}$$

$$< \lim_{t\to\infty} \frac{\sum_{i=1}^t w_i^3 p_i(1-p_i)}{\left(\sum_{i=1}^t w_i^2 p_i(1-p_i)\right)^{3/2}}.$$

Furthermore, we know that each w_i is an integer less than or equal to $C(n)$. Therefore, we have that

$$\lim_{t\to\infty} \frac{\sum_{i=1}^t w_i^3 p_i(1-p_i)}{\left(\sum_{i=1}^t w_i^2 p_i(1-p_i)\right)^{3/2}} \le \lim_{t\to\infty} \frac{C(n)\sum_{i=1}^t w_i^2 p_i(1-p_i)}{\left(\sum_{i=1}^t w_i^2 p_i(1-p_i)\right)^{3/2}}$$
$$= \lim_{t\to\infty} \frac{C(n)}{\left(\sum_{i=1}^t w_i p_i(1-p_i)\right)^{1/2}},$$

which goes to 0 when $\left(\sum_{i=1}^t w_i p_i(1-p_i)\right)^{1/2}$ grows asymptotically more quickly than $C(n)$ as t (and therefore n) grows. Indeed, note that because $p_i \in [\beta, 1-\beta]$ and $w_i \in \mathbb{Z}^+$, we know that

$$w_i p_i(1-p_i) \ge p_i(1-p_i) \ge \beta(1-\beta).$$

Therefore,

$$\begin{aligned}\left(\sum_{i=1}^t w_i p_i(1-p_i)\right)^{1/2} &\ge \left(\sum_{i=1}^t p_i(1-p_i)\right)^{1/2}\\ &\ge (\beta(1-\beta)t)^{1/2}\\ &\ge \left(\beta(1-\beta)\frac{n}{C(n)}\right)^{1/2}\\ &\in \omega(C(n)),\end{aligned}$$

where the third transition follows from $t \ge n/C(n)$, and the last from $C(n) \in o(n^{1/3})$. This concludes the proof. □

We also require the following simple lemma.

Lemma 4. *Given a normally distributed variable $X \sim \mathcal{N}(\mathbb{E}[X], \mathrm{Var}[X])$ with $\mathbb{E}[x] \in [\mu_{min}, \mu_{max}]$ and $\mathrm{Var}[X] \in [\sigma_{min}^2, \sigma_{max}^2]$, then the following is true.*

Case 1: if $\mu_{max} > k$:

$$\Pr[X > k] \le \Pr[Y \sim \mathcal{N}(\mu_{max}, \sigma_{min}^2) > k]$$
$$\Pr[X > k] \ge \Pr[Y \sim \mathcal{N}(\mu_{min}, \sigma_{max}^2) > k]$$

Case 2: if $\mu_{max} < k$:

$$\Pr[X > k] \le \Pr[Y \sim \mathcal{N}(\mu_{max}, \sigma_{max}^2) > k]$$
$$\Pr[X > k] \ge \Pr[Y \sim \mathcal{N}(\mu_{min}, \sigma_{min}^2) > k]$$

Proof. For both upper bounds, we want to minimize the value of $\Phi\left(\frac{k-\mathbb{E}[X]}{\mathrm{Var}[X]}\right)$. Because Φ is monotonically increasing, this is equivalent to minimizing the value of $\frac{k-\mathbb{E}[X]}{\mathrm{Var}[X]}$. It is clear that $k-\mu_{max} < k-\mu_{min}$. Now, if $k-\mu_{max} < 0$, then

$$\frac{k-\mu_{max}}{\sigma_{min}} < \frac{k-\mu_{max}}{\sigma_{max}}.$$

However, if $k-\mu_{max} > 0$, then

$$\frac{k-\mu_{max}}{\sigma_{max}} < \frac{k-\mu_{max}}{\sigma_{min}}.$$

For both lower bounds, we want to maximize the value of $\Phi\left(\frac{k-\mathbb{E}[X]}{\mathrm{Var}[X]}\right)$. Because Φ is monotonically increasing, this is equivalent to maximizing the value of $\frac{k-\mathbb{E}[X]}{\mathrm{Var}[X]}$. As in the above case, it is clear that $k-\mu_{min} > k-\mu_{max}$. Now, if $k-\mu_{min} < 0$, then

$$\frac{k-\mu_{min}}{\sigma_{max}} > \frac{k-\mu_{min}}{\sigma_{min}}.$$

However, if $k-\mu_{min} > 0$, then

$$\frac{k-\mu_{min}}{\sigma_{min}} > \frac{k-\mu_{min}}{\sigma_{max}}.$$

□

Finally, by definition, $\mathrm{Erf}(\infty) = 1$ and $\mathrm{Erf}(-\infty) = -1$, where $\mathrm{Erf}(\cdot)$ denotes the (Gauss) error function. We will use this fact repeatedly throughout the proof of the theorem.

Proof of Theorem 2. Let us define two random variables, X_D and X_M, where X_D denotes the number of correct votes under the direct voting mechanism D, and X_M represents the number of correct votes under GreedyCap. We are interested in comparing $P_D = \Pr[X_D > n/2]$ and $P_M = \Pr[X_M > n/2]$.

Note that $X_D = \sum_{i=1}^n V_i$, where V_i is the Bernoulli variable representing the vote of voter i. Similarly, $X_M = \sum_{i=1}^t w_i V_i$, where $w_i \in \mathbb{Z}^+$ is the total weight accumulated by each voter who actually casts a vote. Note that each voter cannot accumulate weight greater than $C(n)$, and therefore $w_i \le C(n)$ and $t \ge n/C(n)$. By Lemma 3, we can treat both X_D and X_M as being normally distributed, which means we can use the following formulas.

$$P_D = \int_{n/2}^n \frac{1}{\sqrt{2\pi\,\mathrm{Var}[X_D]}} \exp\left(\frac{-(x-\mathbb{E}[X_D])^2}{2\,\mathrm{Var}[X_D]}\right) dx \tag{21}$$

$$P_M = \int_{n/2}^n \frac{1}{\sqrt{2\pi\,\mathrm{Var}[X_M]}} \exp\left(\frac{-(x-\mathbb{E}[X_M])^2}{2\,\mathrm{Var}[X_M]}\right) dx \tag{22}$$

Note that, from above, the PG property means that there exists ε such that $P_M - P_D > \varepsilon$ for at least one graph G_n on n vertices for all suitably large n. Similarly, the DNH property corresponds to $P_D - P_M < \varepsilon$ for all graphs G_n on n vertices for suitably large n and all values of ε. We now that these two properties hold.

For the PG property, we construct a simple family of examples where the property is satisfied. Let the social graph G be composed of pairs of nodes with one competent voter and one incompetent voter with an edge pointing to the competent voter. The competent voters have competence $1-\beta$ and the incompetent voters have competence β. If the voters vote independently, the symmetry between the competent and incompetent voters makes it clear that $P_D = 1/2$. Under Algorithm 1, the incompetent voters all delegate to the competent voters. We now have $\frac{n}{2}$ independent voters who each have one vote of weight two and competence $1-\beta$. By the Condorcet Jury Theorem (Grofman, Owen, and Feld 1983), it follows that P_M approaches 1.

In the remainder of the proof, therefore, we focus on establishing the DNH property. We first show that

$$\mathrm{Var}[X_D] \in [\beta(1-\beta)n, n/4]. \tag{23}$$

Indeed, $X_D = \sum_{i=1}^n V_i$, where V_i is the Bernoulli random variable representing the vote of voter i. In particular, $V_i \sim \text{Bernoulli}(p_i)$, where $p_i \in [\beta, 1-\beta]$ is the competence level of voter i. Because all voters vote independently, $\mathrm{Var}[X_D] = \sum_{i=1}^n \mathrm{Var}[V_i]$, and

$$\mathrm{Var}[V_i] = p_i(1-p_i) \in [\beta(1-\beta), (1/2)^2].$$

This establishes Equation (23).

Now, let us separate the instances into two cases:

1. $|\mathbb{E}[X_D] - \frac{n}{2}| > \frac{n}{\log n}$.
2. $|\mathbb{E}[X_D] - \frac{n}{2}| \leq \frac{n}{\log n}$.

Case 1. In this case, we can give strong lower bounds on both P_D and P_M.

Subcase 1: $\mathbb{E}[X_D] < n/2 - n/\log n$. By Equation (23), $\mathrm{Var}[X_D] \leq n/4 < n$. Because $\mathbb{E}[X_D] < n/2$, by Lemma 4 we have

$$P_D < \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi n^2}} e^{\frac{-\left(x-\frac{n}{2}+\frac{n}{\log n}\right)^2}{2n}}\, dx. \tag{24}$$

This is equivalent to

$$P_D < \frac{1}{2}\left(\text{Erf}\left(\frac{\sqrt{n}(2+\log n)}{2\sqrt{2}\log n}\right) - \text{Erf}\left(\frac{\sqrt{n}}{\sqrt{2}\log n}\right)\right).$$

As n approaches infinity, both arguments go to infinity, and therefore (as $\text{Erf}\,(\infty) = 1$) P_D approaches 0. This means that, no matter the value of P_M, DNH is satisfied.

Subcase 2: $\mathbb{E}[X_D] > n/2 + n/\log n$. We now examine the maximum possible value of $\mathrm{Var}[X_M] = \sum_{i=1}^n w_i^2 \,\mathrm{Var}[V_i]$, where w_i is the total weight accumulated by voter i and, again, V_i is the Bernoulli random variable representing the vote of voter i. Note that here, unlike in Lemma 3, it is possible for w_i to be 0. Additionally, $\mathrm{Var}[V_i] \in [\beta(1-\beta), 1/4]$, and applying this yields $\mathrm{Var}[X_M] \leq (1/4)\cdot\sum_{i=1}^n w_i^2$. Because each voter can accumulate at most weight $C(n)$, by the convexity of x^2, we can see that this is maximized when the maximum number of voters have weight exactly $C(n)$. Therefore, we have

$$\mathrm{Var}[X_M] \leq \frac{1}{4}\cdot \sum_{i=1}^{\lceil n/C(n)\rceil} C(n)^2 < nC(n).$$

Because $\mathbb{E}[X_D] > n/2$, by Lemma 4 we have

$$P_M > \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi nC(n)}} e^{\frac{-\left(x-\frac{n}{2}+\frac{n}{\log n}\right)^2}{2nC(n)}}\, dx. \tag{25}$$

This simplifies to

$$P_M > \frac{1}{2}\left(\text{Erf}\left(\frac{\sqrt{n}(\log n - 2)}{2\sqrt{2C(n)}\log n}\right) + \text{Erf}\left(\frac{\sqrt{n}}{\sqrt{2C(n)}\log n}\right)\right).$$

As n approaches infinity, both arguments go to infinity, and P_M approaches 1. Therefore, no matter what the value of P_D, DNH is satisfied.

Case 2. In this case, we split the argument into two further subcases:

1. The number of voters who delegate is greater than $n/g(n)$, where $g(n)$ is $o(\log n)$ and $\omega(C(n)^2)$.
2. The number of voters who delegate is less or equal to $n/g(n)$.

Subcase 1: Due to delegation, we have $\mathbb{E}[X_M] - \mathbb{E}[X_D] \geq \alpha n/g(n)$. We can now bound the mean by

$$\mathbb{E}[X_M] \geq \frac{n}{2} - \frac{n}{\log n} + \frac{n\alpha}{g(n)}.$$

Therefore, because $g(n) = o(\log n)$, $\mathbb{E}[X_M] > n/2$ as n increases. As before, we also know that $\mathrm{Var}[X_M]$ is bounded from above by $nC(n)$, and therefore, by Lemma 4,

$$P_M \geq \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi nC(n)}} e^{\frac{-\left(x-\frac{n}{2}-\frac{n}{\log n}-\frac{n\alpha}{g(n)}\right)^2}{2nC(n)}}\, dx. \tag{26}$$

We would like to show that this integral goes to 1 as n goes to infinity. This is equivalent to

$$\frac{1}{2}\left(\text{Erf}\left(\frac{\frac{n}{2}-\frac{n\alpha}{g(n)}+\frac{n}{\log n}}{\sqrt{2nC(n)}}\right) - \text{Erf}\left(\frac{\sqrt{n}\left(-\frac{\alpha}{g(n)}+\frac{1}{\log n}\right)}{\sqrt{2C(n)}}\right)\right).$$

Note that as n goes to infinity, the first argument goes to infinity and the second argument goes to negative infinity when $g(n) = o(\log n)$. Therefore, P_M goes to 1, satisfying DNH.

Subcase 2: In this case, most voters remain independent. We will argue that although the delegation does impact the variance, this impact will get arbitrarily small as n grows larger, implying that the loss will get arbitrarily small.

Let us index the voters according to what happens in the delegation scheme. Let the first n_1 indexed voters represent those who remain independent and do not get delegated a vote. Let the next n_2 indexed voters be those who got delegated at least one vote. Finally, the last $n - n_1 - n_2$ indexed voters are those who delegated their vote to another voter. Based on our assumption above, we know that

$\lim_{n\to\infty} \frac{n_1}{n} = 1$; most voters remain independent and unaffected by the delegation scheme.

Additionally, note that the mean will be slightly different in the two schemes, but this to our advantage because the mean will improve in the delegation scheme due to "uphill" delegation.

Therefore, given

$$P_D = \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi \operatorname{Var}[X_D]}} e^{\frac{-(x-\mathbb{E}[X_D])^2}{2\operatorname{Var}[X_D]}}\, dx$$

and

$$P_M = \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi \operatorname{Var}[X_M]}} e^{\frac{-(x-\mathbb{E}[X_M])^2}{2\operatorname{Var}[X_M]}}\, dx,$$

because $\mathbb{E}[X_M] \geq \mathbb{E}[X_D]$, we can say that

$$P_M \geq \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi \operatorname{Var}[X_M]}} e^{\frac{-(x-\mathbb{E}[X_D])^2}{2\operatorname{Var}[X_M]}}\, dx.$$

Now, we have to relate $\operatorname{Var}[X_M]$ and $\operatorname{Var}[X_D]$. Ideally, we want to show that they are multiplicatively close to each other.

We can decompose the variance of X_d.

$$\operatorname{Var}[X_D] = \sum_{i=1}^{n_1} p_i(1-p_i) + \sum_{i=n_1+1}^{n} p_i(1-p_i).$$

Likewise, we can decompose the variance of X_M.

$$\operatorname{Var}[X_M] = \sum_{i=1}^{n_1} p_i(1-p_i) + \sum_{i=n_1+1}^{n_1+n_2} w_i^2 p_i(1-p_i) + \sum_{i=n_1+n_2+1}^{n} 0.$$

Therefore, we have

$$\begin{aligned}
\operatorname{Var}[X_M] - \operatorname{Var}[X_D] &= \sum_{i=n_1+1}^{n_1+n_2} (w_i^2-1)p_i(1-p_i) \\
&\quad - \sum_{i=n_1+n_2+1}^{n} p_i(1-p_i) \\
&\leq \sum_{i=n_1+1}^{n_1+n_2} (w_i^2-1)p_i(1-p_i) \\
&\leq \frac{n_2}{4}(\max {w_i}^2 - 1) \\
&\leq \frac{1}{4}\cdot\frac{n}{g(n)}(C(n)^2-1),
\end{aligned}$$

where the last inequality holds because $w_i \leq C(n)$, and n_2, the number of voters who are delegated to, is at most the number of voters who delegate, which is at most $n/g(n)$ by assumption.

This means that

$$\operatorname{Var}[X_M] \leq \operatorname{Var}[X_D] + \frac{1}{4}\cdot\frac{n}{g(n)}(C(n)^2-1)$$

and therefore

$$\begin{aligned}
\frac{\operatorname{Var}[X_M]}{\operatorname{Var}[X_D]} &\leq \frac{\operatorname{Var}[X_D] + \frac{1}{4}\cdot\frac{n}{g(n)}(C(n)^2-1)}{\operatorname{Var}[X_D]} \\
&= 1 + \frac{\frac{n}{g(n)}(C(n)^2-1)}{4\operatorname{Var}[X_D]}.
\end{aligned}$$

Now, note that by Equation (23),

$$\operatorname{Var}[X_D] \geq n\beta(1-\beta)$$

and therefore

$$\begin{aligned}
\operatorname{Var}[X_M] &\leq \operatorname{Var}[X_D]\left(1 + \frac{\frac{n}{g(n)}(C(n)^2-1)}{4n\beta(1-\beta)}\right) \\
&= \operatorname{Var}[X_D]\left(1 + \frac{1}{g(n)}\cdot\frac{C(n)^2-1}{4\beta(1-\beta)}\right).
\end{aligned}$$

Let

$$\eta = \frac{1}{g(n)}\cdot\frac{C(n)^2-1}{4\beta(1-\beta)}$$

and note that as n goes to infinity, η goes to 0 because we chose $g(n)$ to grow asymptotically faster than $C(n)^2$.

Therefore, revisiting the original integrals, we have

$$P_D = \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi \operatorname{Var}[X_D]}} e^{\frac{-(x-\mathbb{E}[X_D])^2}{2\operatorname{Var}[X_D]}}\, dx$$

and

$$P_M \geq \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi \operatorname{Var}[X_D](1+\eta)}} e^{\frac{-(x-\mathbb{E}[X_D])^2}{2\operatorname{Var}[X_D](1+\eta)}}\, dx.$$

Simplifying the above yields

$$P_D = \frac{1}{2}\left(\operatorname{Erf}\left(\frac{n-\mathbb{E}[X_D]}{\sqrt{2\operatorname{Var}[X_D]}}\right) - \operatorname{Erf}\left(\frac{n-2\mathbb{E}[X_D]}{2\sqrt{2\operatorname{Var}[X_D]}}\right)\right) \tag{27}$$

and

$$\begin{aligned}
P_M \geq \frac{1}{2}\Bigg(&\operatorname{Erf}\left(\frac{n-\mathbb{E}[X_D]}{\sqrt{2\operatorname{Var}[X_D](1+\eta)}}\right) \\
&-\operatorname{Erf}\left(\frac{n-2\mathbb{E}[X_D]}{2\sqrt{2\operatorname{Var}[X_D](1+\eta)}}\right)\Bigg)
\end{aligned} \tag{28}$$

Furthermore, again by Equation (23), we know that $\operatorname{Var}[X_D] \in [\beta(1-\beta)n, n/4]$ and therefore $\sqrt{\operatorname{Var}[X_D]} = \sqrt{cn}$, where $c \in [\beta(1-\beta), 1/4]$. From this, note that as n goes to infinity, the argument to the first error function in each expression goes to infinity.

Let

$$h_1(n) = \frac{n-2\mathbb{E}[X_D]}{2\sqrt{2\operatorname{Var}[X_D]}} \tag{29}$$

be the argument to the second error function in (27), and let

$$h_2(n) = \frac{n - 2\mathbb{E}[X_D]}{2\sqrt{2\,\mathrm{Var}[X_D](1+\eta)}} \tag{30}$$

be the argument to the second error function in (28). As n goes to infinity, note that the argument to (29) must go to one of four states: infinity, negative infinity, zero, or a constant. In the case that it goes to infinity, negative infinity, or zero, the presence of the extra $\frac{1}{\sqrt{1+\eta}}$ term in (30) does nothing to change the sign of the arguments, and therefore they each converge to the same state (infinity, negative infinity, or zero) as n approaches infinity. When the argument to (29) goes to a constant, note that as n goes to infinity, η goes to 0, and therefore the two converge once again.

We conclude that (an upper bound on) the difference between P_D and P_M converges to 0, and hence DNH is satisfied. $\square$