AdaBoost

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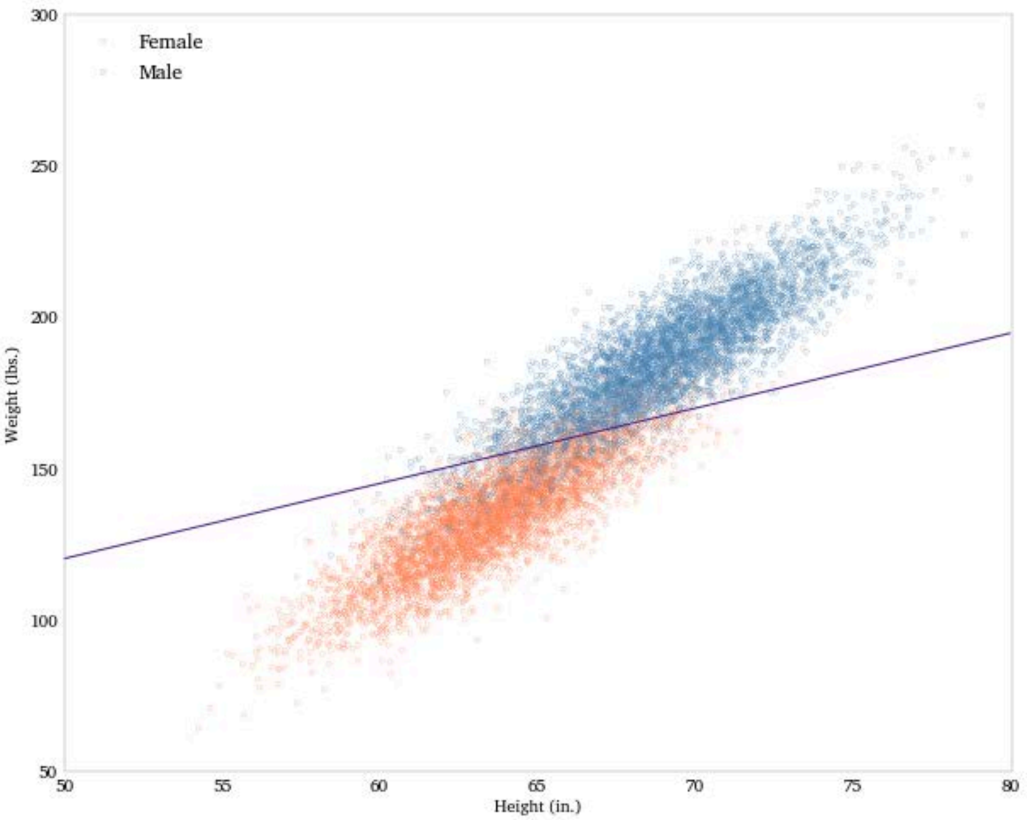
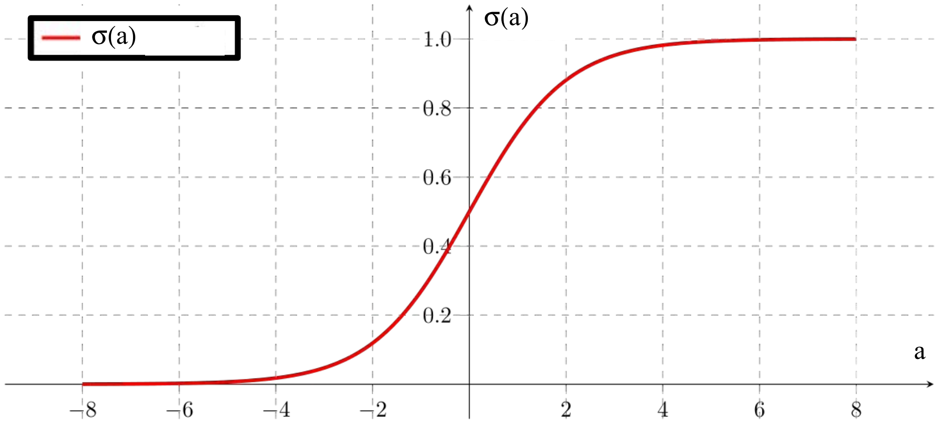
IC-CVLab

Reminder: Logistic Regression



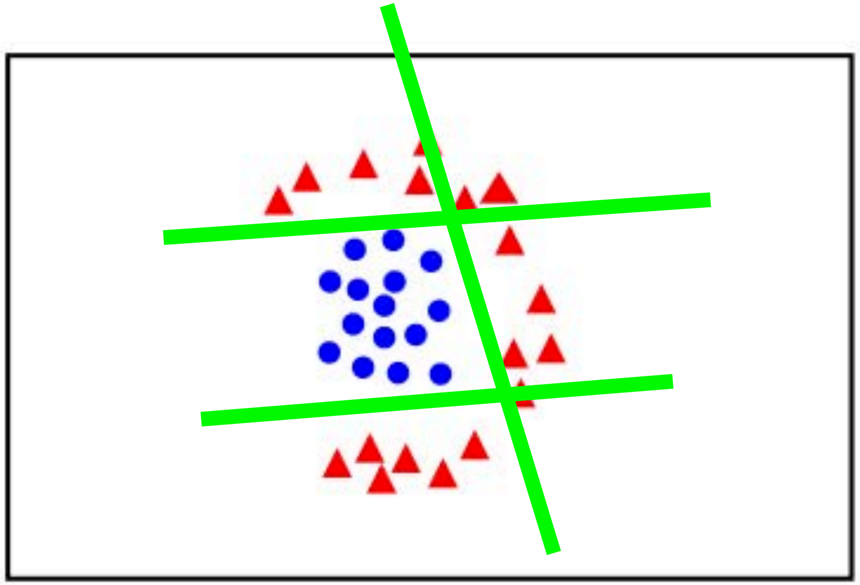
$$
\begin{aligned}
y(\mathbf{x};\mathbf{w},w_0) &= \sigma(\mathbf{w}^T\mathbf{x}+w_0)\\
&\approx p(t=1,\mathbf{x})
\end{aligned}
$$



Given the training set $\{(x_n, t_n)_{1\leq n\leq N}\}$, choose a $\mathbf{w}$ that minimizes

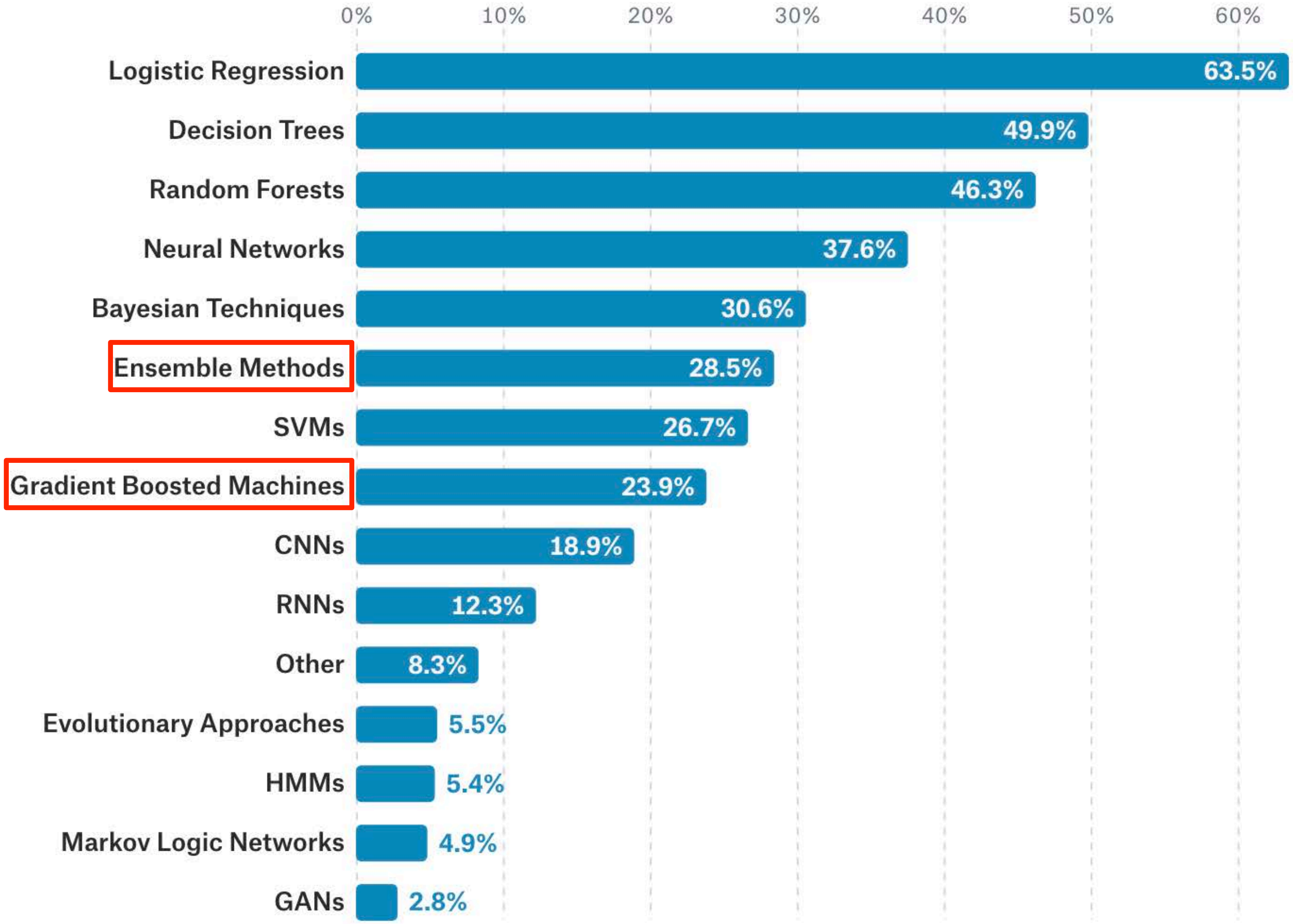
$$
E(\mathbf{w},w_0) = -\sum_n \{t_n \ln y_n + (1-t_n)\ln(1-y_n)\} \approx -\ln(p(\mathbf{t}|\mathbf{w},w_0))\ .
$$

Non Linearly Separable Data

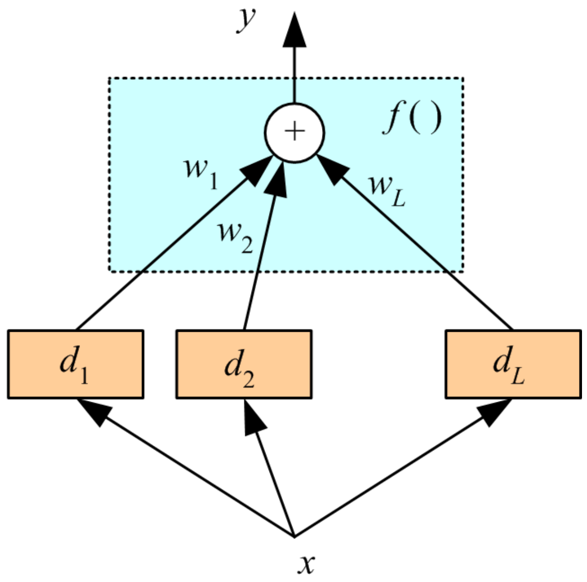


- One approach is to combine multiple linear classifiers.
- We will see other ones in the next classes.

Boosting Methods



Combining Linear Classifiers



- Use the linear classifiers as “weak” classifiers, that is, classifiers operating only slightly better than chance.
- Write a strong classifier as a weighted sum of weak ones.

Ada Boost

Iteratively building a weighted sum of weak classifiers:

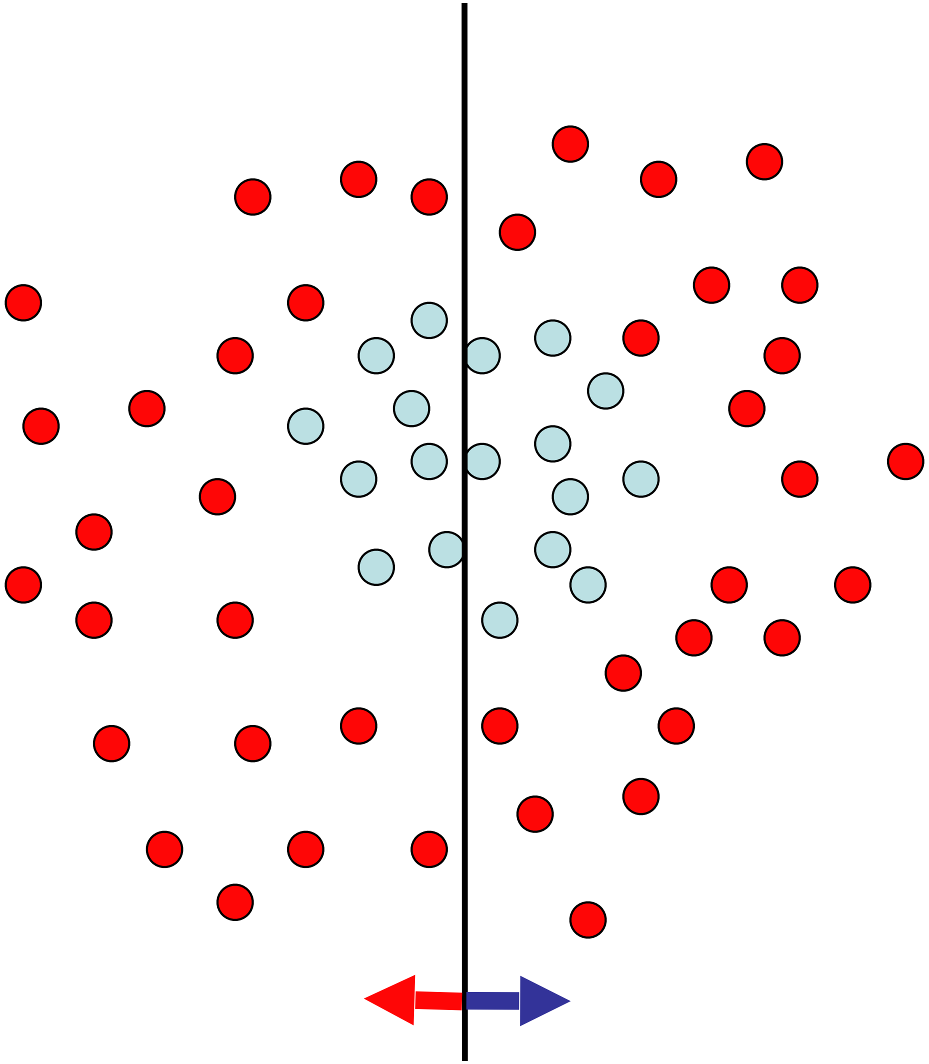
$$y(\mathbf{x}) = \alpha_1 y_1(\mathbf{x}) + \alpha_2 y_2(\mathbf{x}) + \alpha_3 y_3(\mathbf{x}) + \ldots$$

Weak classifier

Strong classifier

Weight

Toy Example



Each data point has a class label:

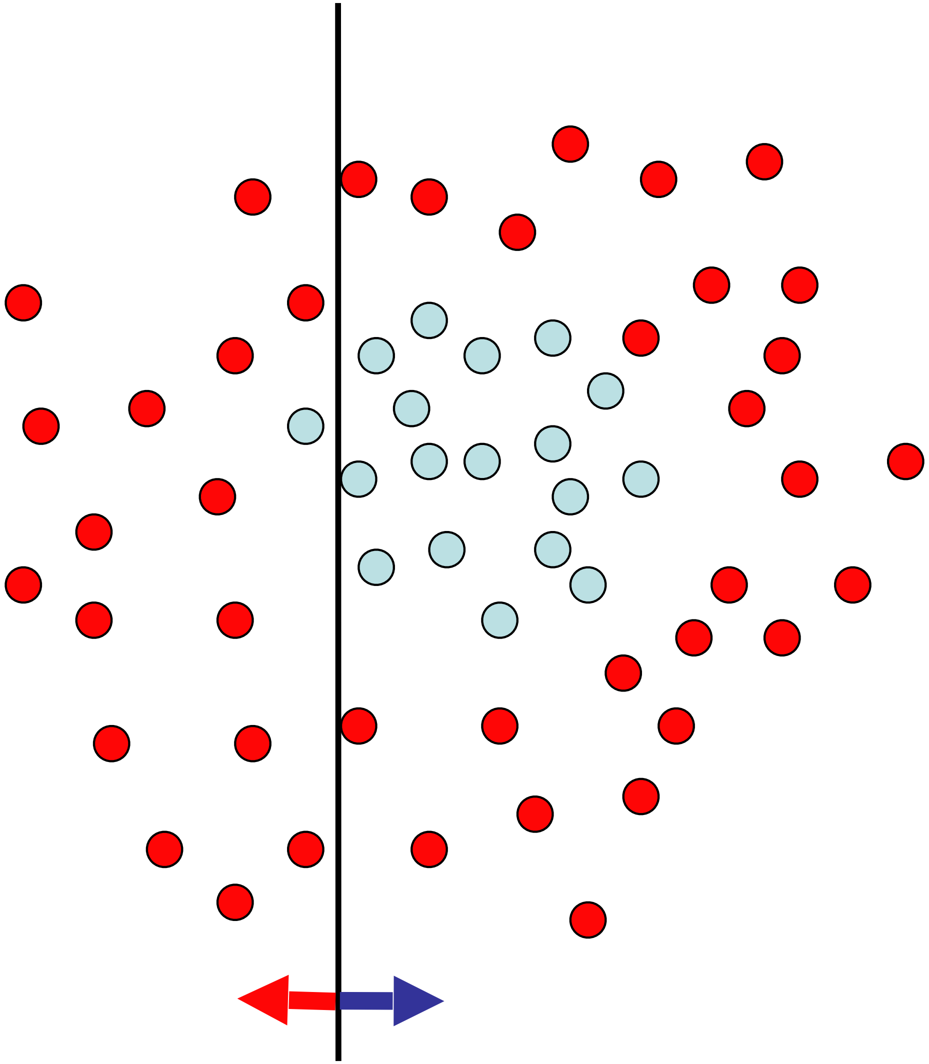
$$y_t = \begin{cases} +1 & (\text{●}) \\ -1 & (\text{○}) \end{cases}$$

and a weight:

$w_t = 1$

Classifier is roughly at chance.

Toy Example



Each data point has

a class label:

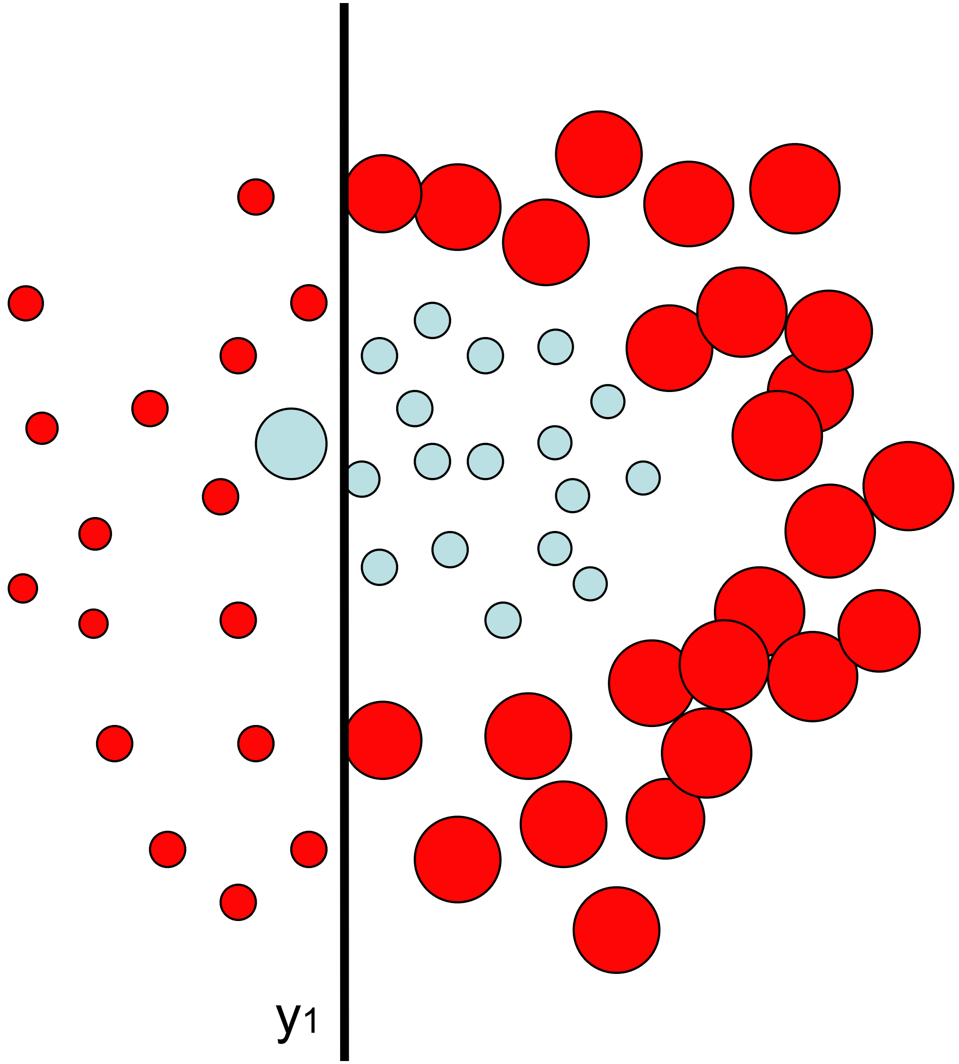
$$y_t = \begin{cases} +1 & (🔴) \\ -1 & (○) \end{cases}$$

and a weight:

$w_t = 1$

Classifier is now slightly better than chance.
It becomes y_1.

Toy Example



Each data point is given a new class label

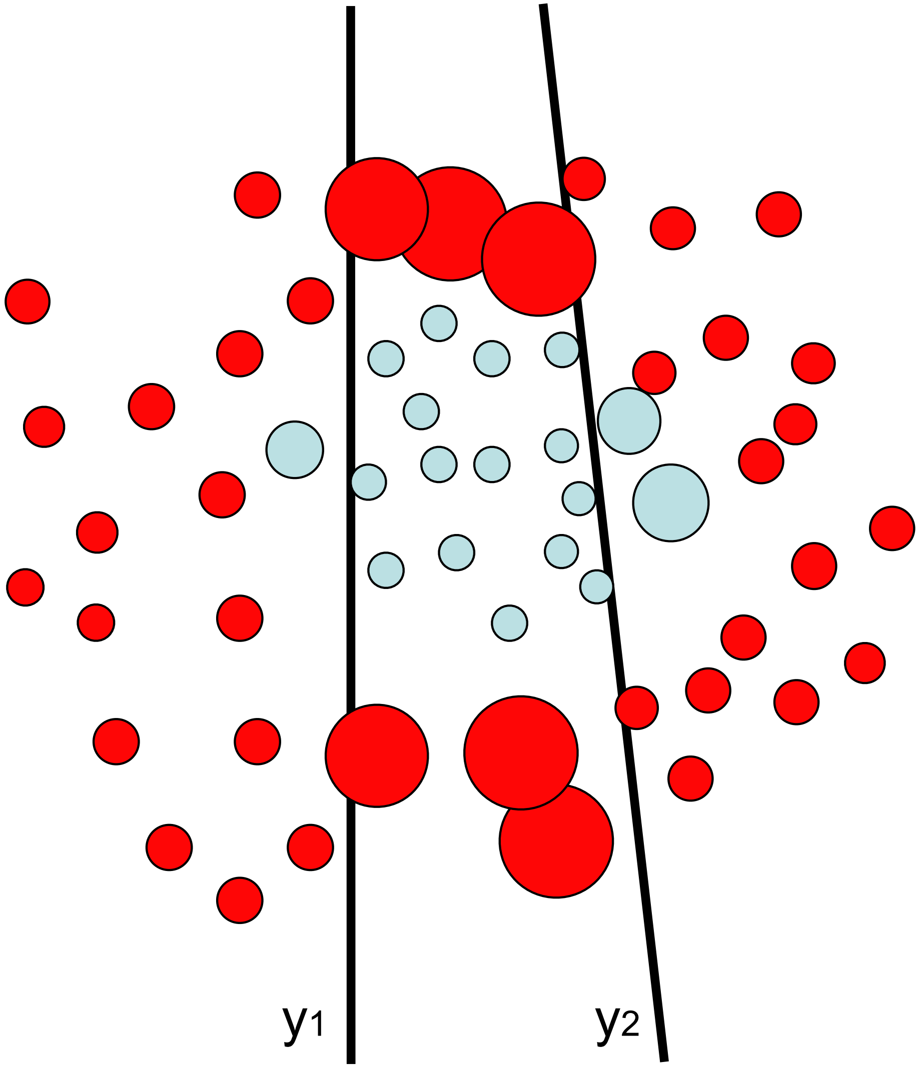
$$y_t = \begin{cases} +1 & (●) \\ -1 & (○) \end{cases}$$

and a new weight

$\mathbf{w}_t$

$\mathbf{w}_t$ is chosen so that the classifier operates at chance again.

Toy Example



Each data point is given a new class label

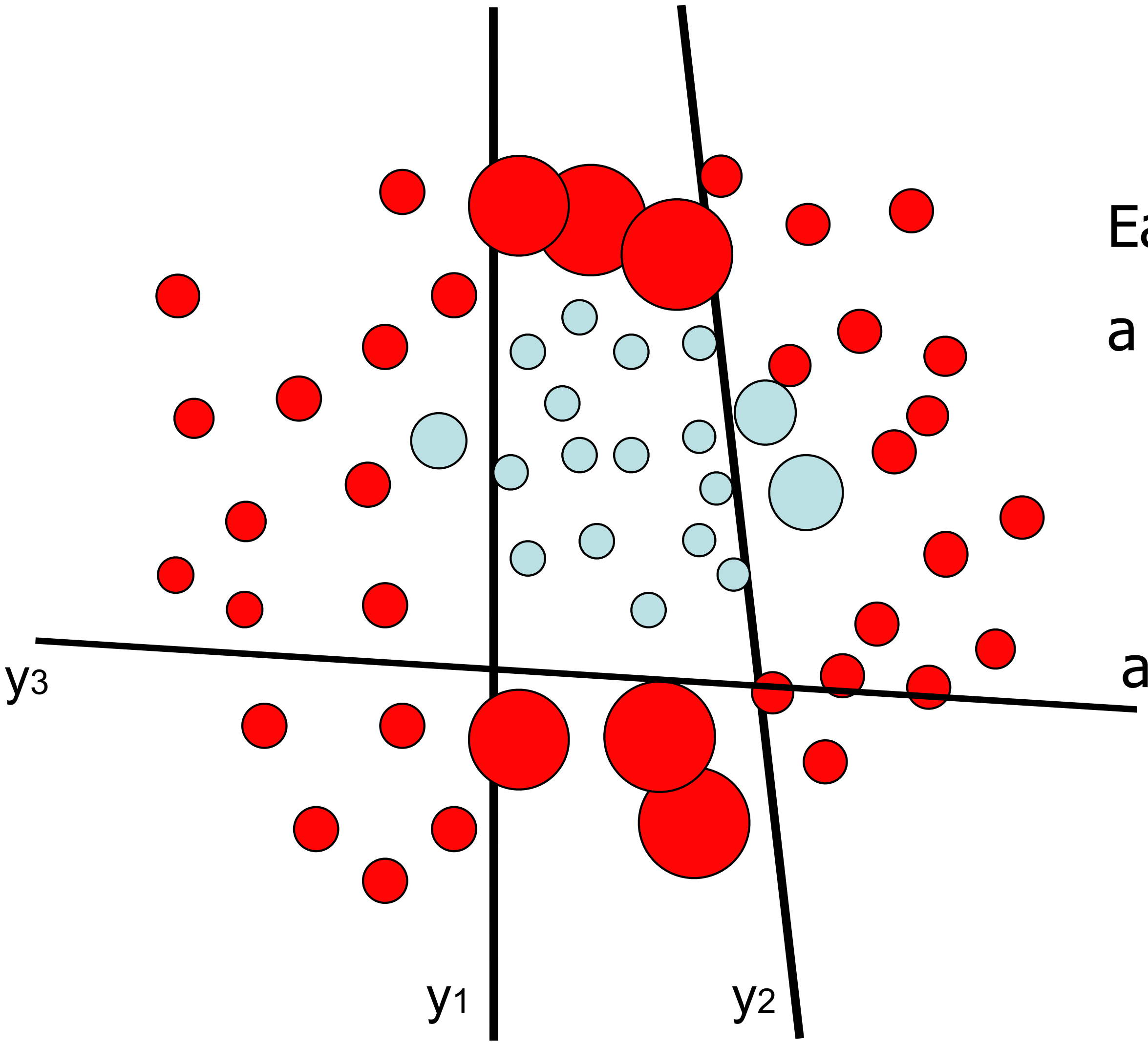
$$y_t = \begin{cases} +1 & \text{(●)} \\ -1 & \text{(○)} \end{cases}$$

and a new weight

$\mathbf{w_t}$

Find a new classifier y_2 and reset the weights again.

Toy Example



Each data point is given a new class label

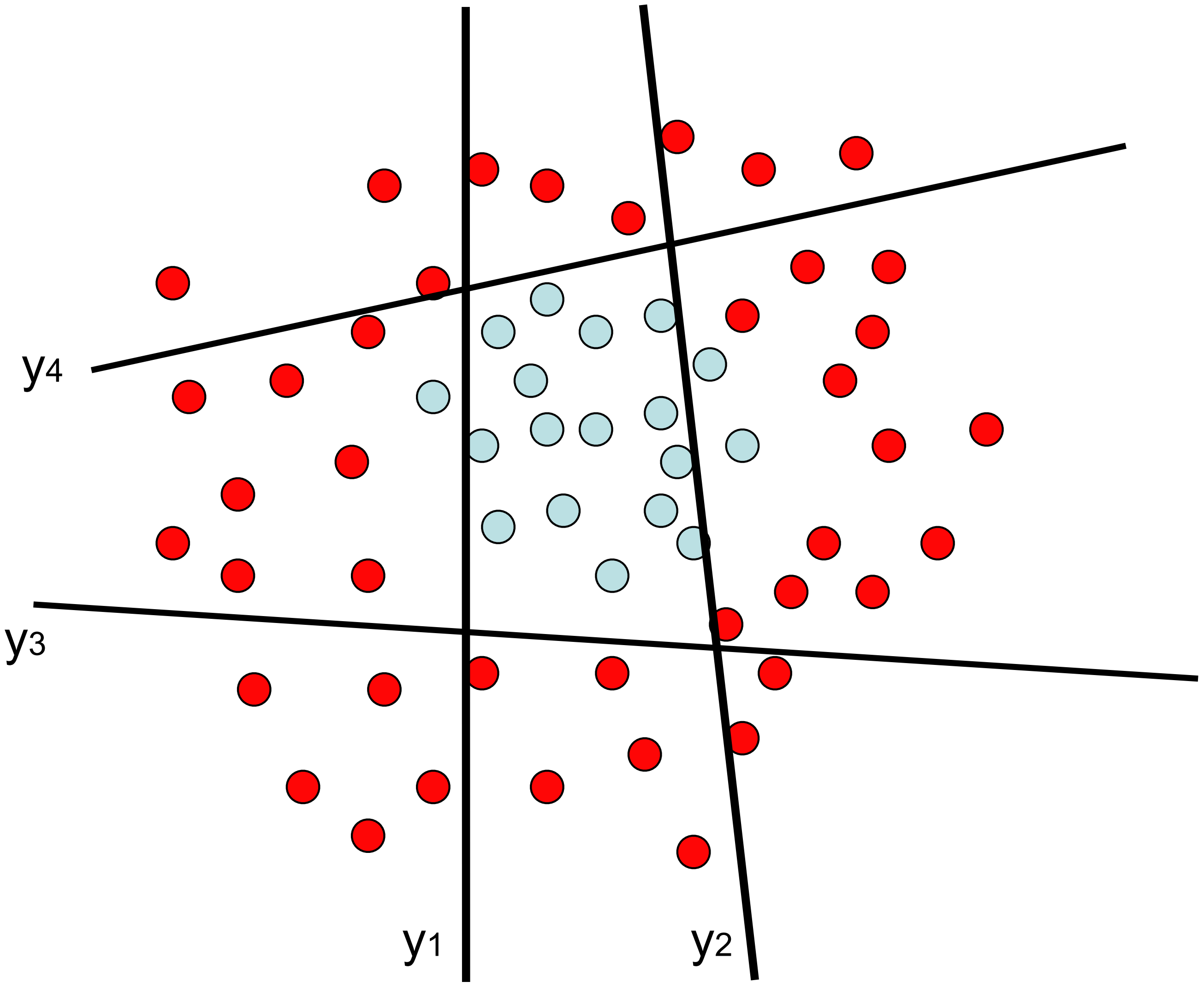
$$y_t = \begin{cases} +1 & (\text{●}) \\ -1 & (\text{○}) \end{cases}$$

and a new weight

$\mathbf{w_t}$

Find a new classifier y_3 and reset the weights again.

Toy Example



$$y(\mathbf{x}) = \alpha_1 y_1(\mathbf{x}) + \alpha_2 y_2(\mathbf{x}) + \alpha_3 y_3(\mathbf{x}) + \alpha_4 y_4(\mathbf{x})$$

Adaboost Algorithm

For a training set $\chi = \{\mathbf{x}_n, t_n\}$ where $t_n \in$ {-1,1} for $1 \leq n \leq$N:

1. Initialize data weights: $\forall n, w_n^1 = 1/N$.

2. For $t = [1, \ldots, T]$:

   (a) Find classifier $y_t : \chi \rightarrow \{-1, 1\}$ that minimizes weighted error $\sum_{t_n \neq y_t(\mathbf{x}_n)} w_n^t$.

   (b) Evaluate

$$\epsilon_t = \frac{\sum_{t_n \neq y_t(\mathbf{x}_n)} w_n^t}{\sum_{n=1}^{N} w_n^t}$$

Inferior to 0.5 if y$_t$ operates at better than chance.

$$\alpha_t = \log(\frac{1-\epsilon_t}{\epsilon_t})$$

Positive if y$_t$ operates at better than chance.

   (c) Update weights

$$w_n^{t+1} = w_n^t \exp(\alpha_t I(t_n \neq y_t(\mathbf{x}_n)))$$

The weight of misclassified samples is increased.

$\rightarrow$ **Final classifier:** $Y(\mathbf{x}) = \text{sign}(\sum_{t=1}^{T} \alpha_t y_t(\mathbf{x}))$

Optional: Proof Sketch (1)

At iteration t:

$$f_t(\mathbf{x}) = \frac{1}{2}\sum_{s=1}^{t} \alpha_s y_s(\mathbf{x}) \ ,$$

$$= f_{t-1}(\mathbf{x}) + \frac{1}{2}\alpha_t y_t(\mathbf{x}) \ .$$

Computed at previous iteration.

To be estimated.

To estimate the unknowns, we seek to minimize

$$E_t = \sum_{n=1}^{N} \exp(-t_n f_t(\mathbf{x}_n)) \ ,$$

Exponential loss.

with respect to α_t and y_t .

Optional: Proof Sketch (2)

At iteration t, given $y_1, \dots, y_{t-1}$ and $\alpha_1, \dots, \alpha_{t-1}$, minimize

$$\begin{aligned} E_t &= \sum_{n=1}^{N} \boxed{\exp(-t_n(f_{t-1}(\mathbf{x}_n)} + \frac{1}{2}\alpha_t y_t(\mathbf{x}_n)))) \\ &= \sum_{n=1}^{N} w_n^t \exp(-\frac{\alpha_t}{2} t_n y_t(\mathbf{x}_n)) \end{aligned}$$

with respect to y_t and α_t.

Optional: Proof (2)

$$
\begin{aligned}
E_t &= \sum_n w_n \exp\Big(-\frac{\alpha_t}{2} t_n y_t(\mathbf{x}_n)\Big) \\
&= \exp(-\alpha_t/2) \sum_{t_n = y_t(\mathbf{x}_n)} w_n + \exp(\alpha_t/2) \sum_{t_n \neq y_t(\mathbf{x}_n)} w_n \\
&= \underbrace{(\exp(\alpha_t/2) - \exp(-\alpha_t/2))}_{\text{Greater than 0}} \sum_{n=1}^{N} \underbrace{I(t_n \neq y_t(\mathbf{x}_n)) w_m^t}_{\text{Must be minimized}} + \exp(-\alpha_t/2) \underbrace{\sum_{n=1}^{N} w_n^t}_{\text{Independent of } y_t}
\end{aligned}
$$

Therefore, for E_t to be minimized, y_t must minimize $\sum_{t_n \neq y_t(\mathbf{x}_n)} w_n^t$.

Optional: Proof (3)

$$E_t = \sum_n w_n \exp(-\frac{\alpha_t}{2} t_n y_t(\mathbf{x}_n))$$

$$\Rightarrow 2\frac{\delta E_t}{\delta \alpha_t} = -\exp(-\alpha_t/2) \sum_{t_n = y_t(\mathbf{x}_n)} w_n + \exp(\alpha_t/2) \sum_{t_n \neq y_t(\mathbf{x}_n)} w_n$$

Therefore:

$$\frac{\delta E_t}{\delta \alpha_t} = 0 \Rightarrow \alpha_t = \log\left[\frac{\sum_{t_n = y_t(\mathbf{x}_n)} w_n}{\sum_{t_n \neq y_t(\mathbf{x}_n)} w_n}\right]$$

$$= \log(\frac{1-\epsilon_t}{\epsilon_t}) \quad \text{with} \quad \epsilon_t = \frac{\sum_{y_t(\mathbf{x}_n) \neq t_n} w_n}{\sum_{n=1}^{N} w_n}$$

Optional: Proof (4)

At the following iteration:

$$\begin{aligned} w_n^{t+1} &= \exp(-t_n f_t(\mathbf{x}_n)) \\ &= \exp(-t_n f_{t-1}(\mathbf{x}_n) - \frac{1}{2}\alpha_t t_n y_t(\mathbf{x}_n)) \\ &= w_n \exp(-\frac{1}{2}\alpha_t t_n y_t(\mathbf{x}_n)) \\ t_n y_t(\mathbf{x}_n) &= 1 - 2I(y_t(\mathbf{x}_n) \neq t_n) \end{aligned}$$

$$\begin{aligned} \Rightarrow w_n^{t+1} &= w_n^t \exp(-\alpha_t/2) \exp(\alpha_t I(t_n \neq y_t(\mathbf{x}_n))) \\ &\propto w_n^t \exp(\alpha_t I(t_n \neq y_t(\mathbf{x}_n))) \end{aligned}$$

Optional: Proof Sketch (5)

Minimizing $\sum_{n=1}^{N} w_n^t \exp(-\frac{\alpha_t}{2} t_n y_t(\mathbf{x}_n))$ w.r.t. to y_t and α_t yields:

$$y_t \quad \text{must minimize} \quad \sum_{t_n \neq y_t(\mathbf{x}_n)} w_n^t$$

$$\alpha_t = \log(\frac{1-\epsilon_t}{\epsilon_t}) \text{ with } \epsilon_t = \frac{\sum_{t_n \neq y_t(\mathbf{x}_n)} w_n^t}{\sum_{n=1}^{N} w_n^t}$$

$$w_n^{t+1} = w_n^t \exp(\alpha_t I(t_n \neq y_t(\mathbf{x}_n)))$$

—> Adaboost performs a form of gradient descent on the exponential loss.

Adaboost Algorithm

For a training set $\chi = \{\mathbf{x}_n, t_n\}$ where $t_n \in$ {-1,1} for 1 $\leq n \leq$N:

1. Initialize data weights: $\forall n, w_n^1 = 1/N$.
2. For $t = [1, \ldots, T]$:
   (a) Find classifier $y_t : \chi \rightarrow \{-1, 1\}$ that minimizes weighted error $\sum_{t_n \neq y_t(\mathbf{x}_n)} w_n^t$.
   (b) Evaluate

$$\epsilon_t = \frac{\sum_{t_n \neq y_t(\mathbf{x}_n)} w_n^t}{\sum_{n=1}^{N} w_n^t}$$

$$\alpha_t = \log(\frac{1 - \epsilon_t}{\epsilon_t})$$

   (c) Update weights

$$w_n^{t+1} = w_n^t \exp(\alpha_t I(t_n \neq y_t(\mathbf{x}_n)))$$

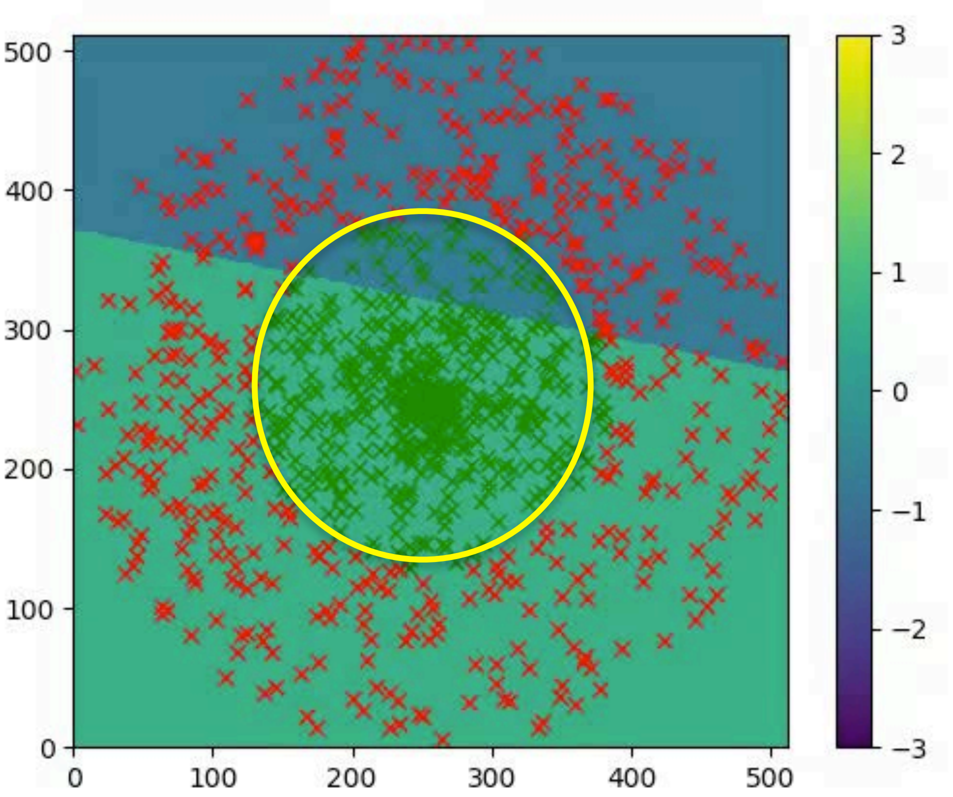
$\rightarrow$ **Final classifier:** $Y(\mathbf{x}) = \text{sign}(\sum_{t=1}^{T} \alpha_t y_t(\mathbf{x}))$

Adabost in Python

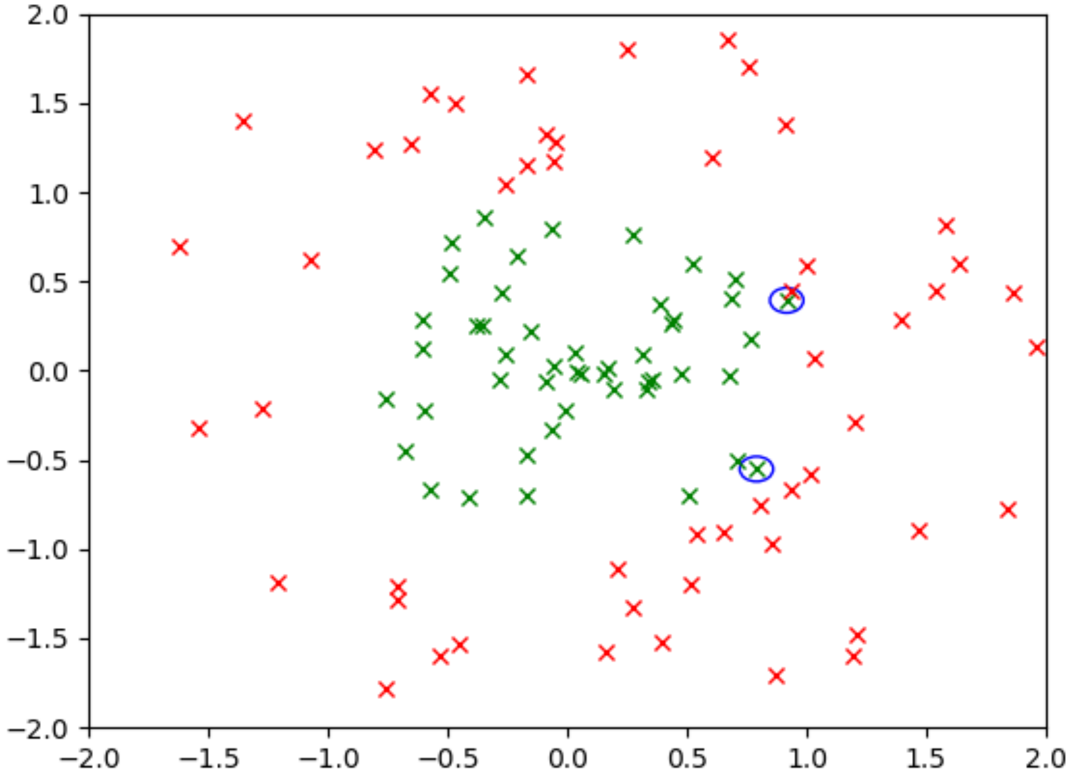
```
def fit(self,nit=10):
    # Initialize weights and list of classifiers
     self.weakCls  = []
     bestAcc = 0.0
     self.datCoeffs = np.ones(self.ns,dtype=np.float)/self.ns
    # Find nit weak classifiers and update weights each time.
    for m in range(nit):
        weakC=self.getWeakC()
        self.weakCls.append(weakC)
        weakC.alpha=self.updateWeights(weakC)
```

```
def updateWeights(self,weakC):
    # Compute alpha
    err,_ = self.weakClassError(weakC)
    alpha = np.log(1.0/max(1e-10,err)-1.0)
    # Compute numbers of misclassified samples.
    nerrs = np.logical_not(weakC.predict(self.xs)==self.ys)
    # Update and normalize weights.
    self.datCoeffs *=  np.exp(alpha*nerrs)
    self.datCoeffs /=  sum (self.datCoeffs)
    return alpha
```

Circular Distribution

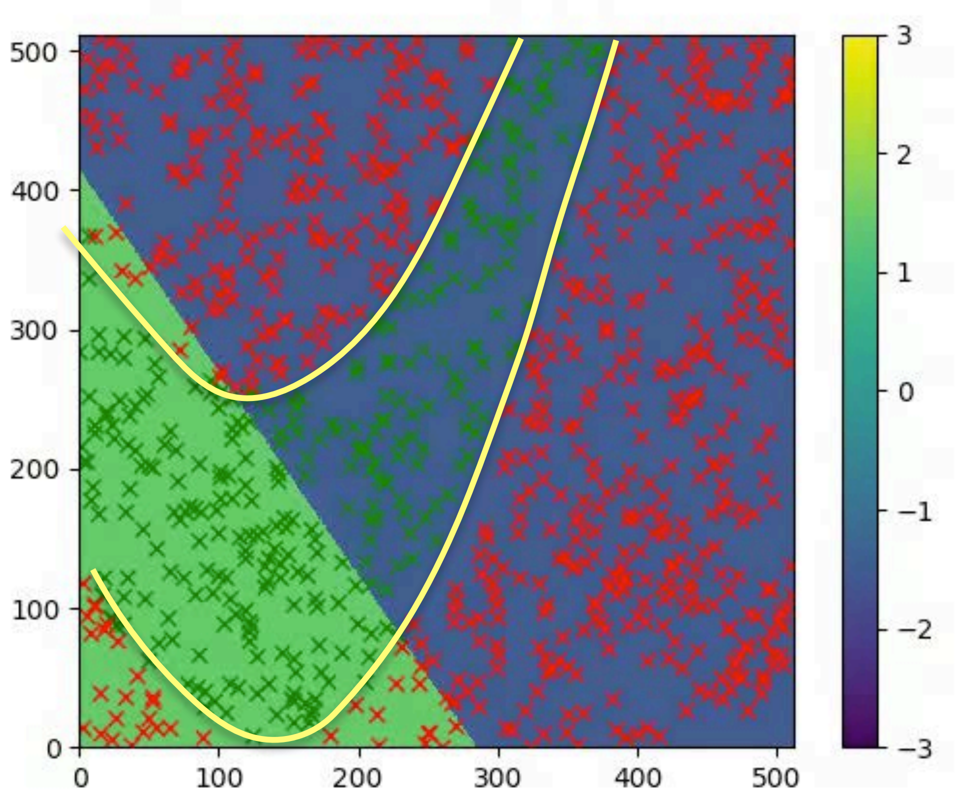


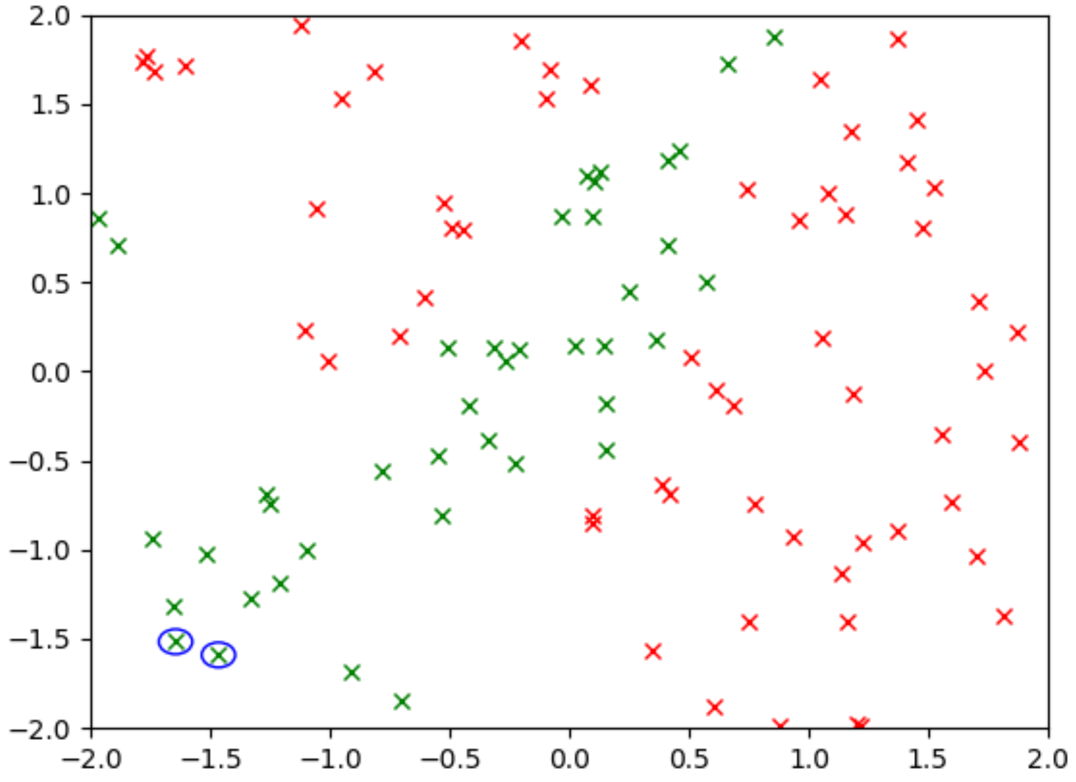
Training (100 iterations)



Validation (98% accuracy)

Rosenbrock Function





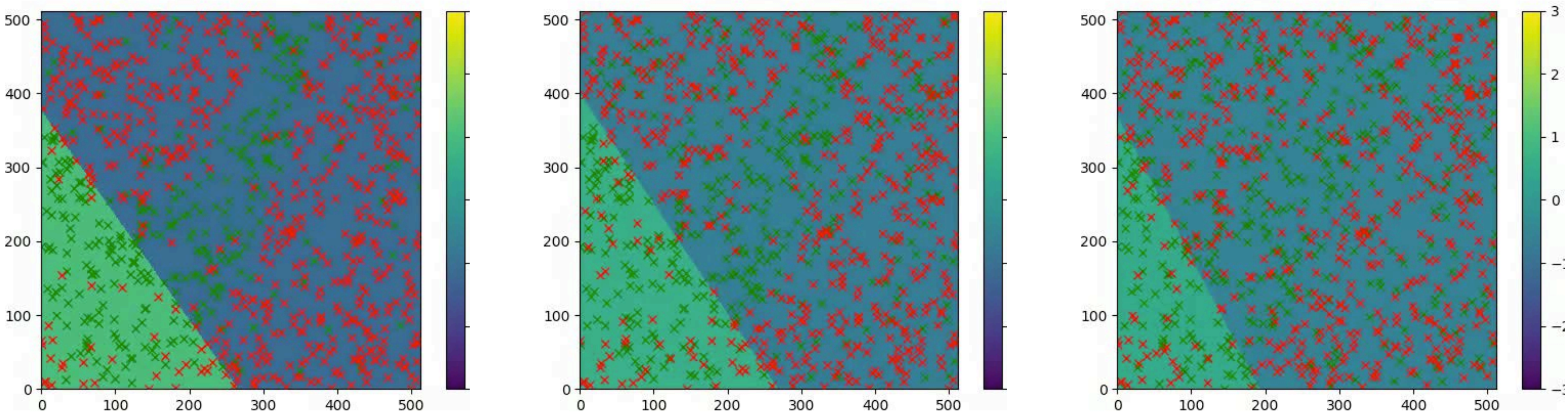
Training (100 iterations)

Validation (98% accuracy)

$$r(x,y) = 100 * (y - x^2)^2 + (1-x)^2$$

$$f(x,y) = \begin{cases} \text{-1} & \text{if } r(x,y) < \mathrm{T} \\ 1 & \text{otherwise} \end{cases}$$

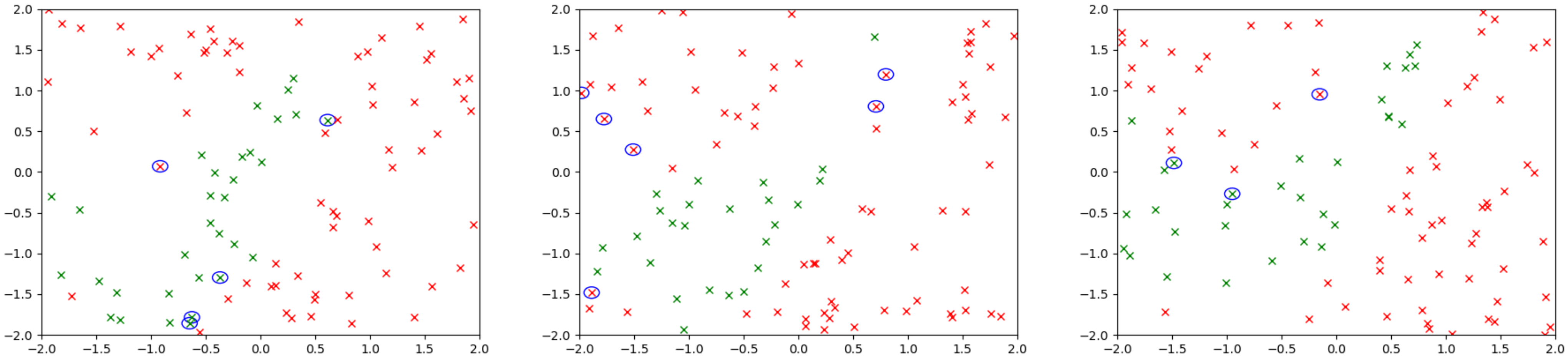
Noisy Labels



10% mislabeled

20% mislabeled

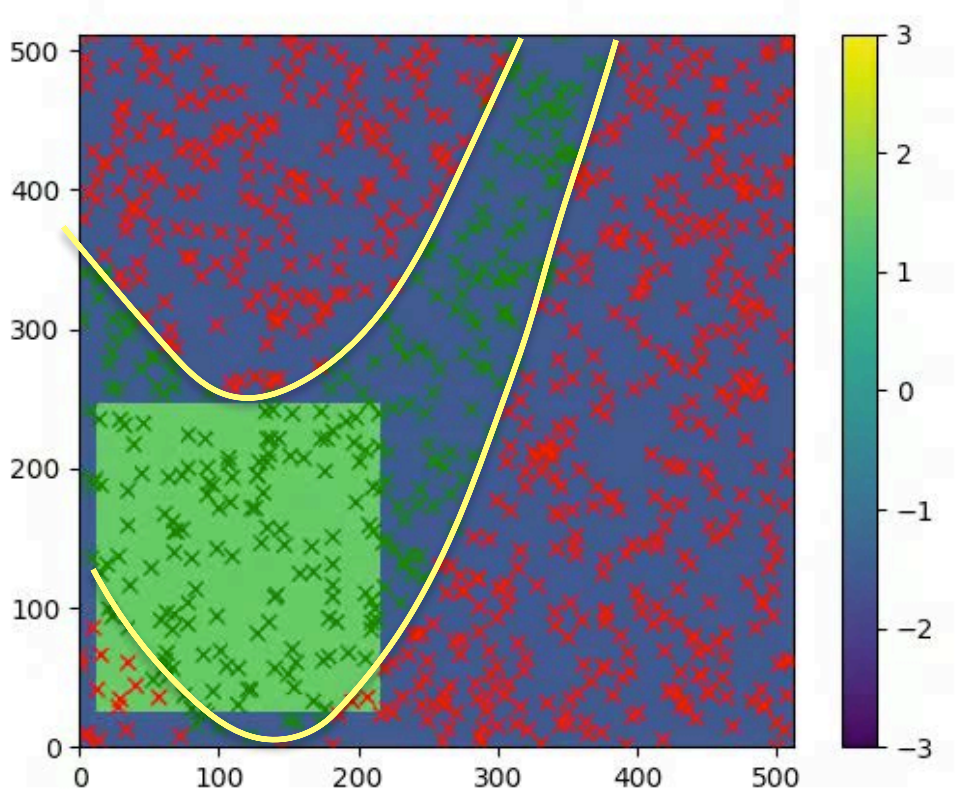
30% mislabeled



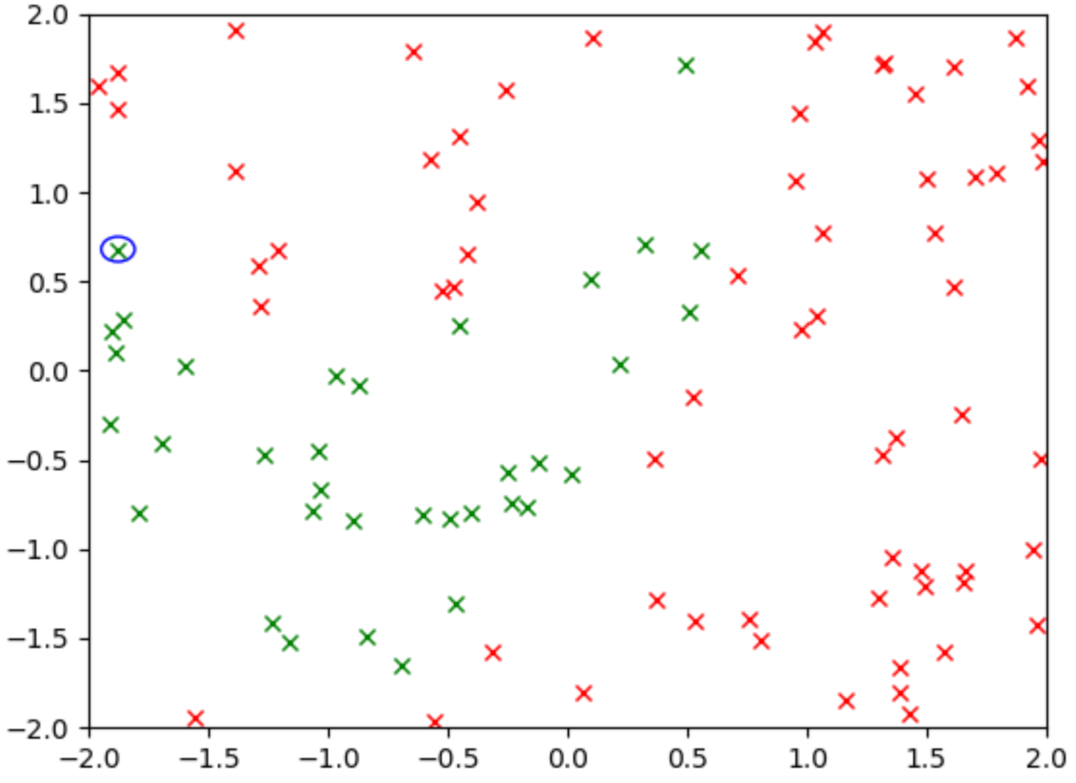
The incorrect labels have relatively little impact because they are randomly distributed, but they could.

Changing the Weak Learners

Using boxes instead of lines.



Training (100 iterations)



Validation (99% accuracy)

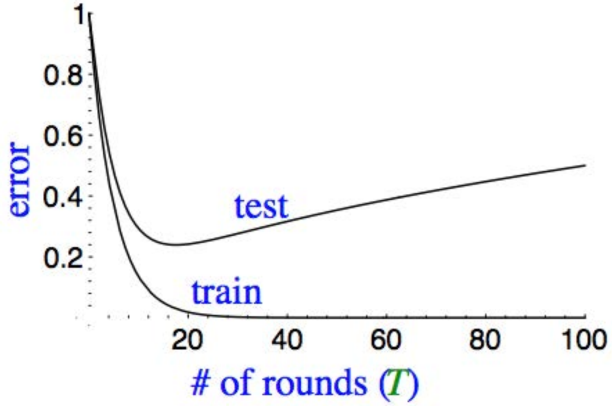
$$
\begin{aligned}
y(\mathbf{x};\mathbf{w}) &= \begin{cases} 1 & \text{if} \quad x_0 < \mathbf{x}[1] < x_1 \quad \text{and} \quad y_0 < \mathbf{x}[2] < y_1 \ , \\ -1 & \text{otherwise.} \end{cases} \\
\mathbf{w} &= (x_0, y_0, x_1, y_1)
\end{aligned}
$$

Training and Testing Errors

- The training error goes down exponentially fast if the weighted errors εt of the component classifiers is always strictly inferior to 0.5.

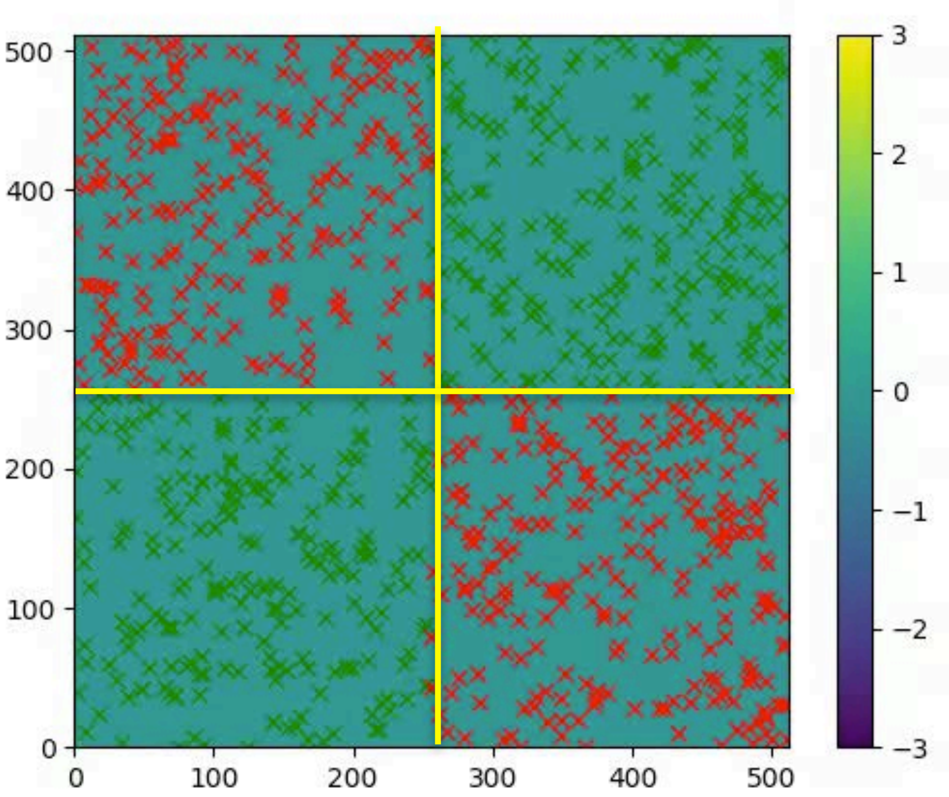
$$\frac{1}{N}\sum_{n}[t_n \neq h(\mathbf{x}_n)] < \prod_{t=1}^{T}\sqrt{\epsilon_t(1-\epsilon_t)}$$

- The testing error may eventually go up due to overfitting.

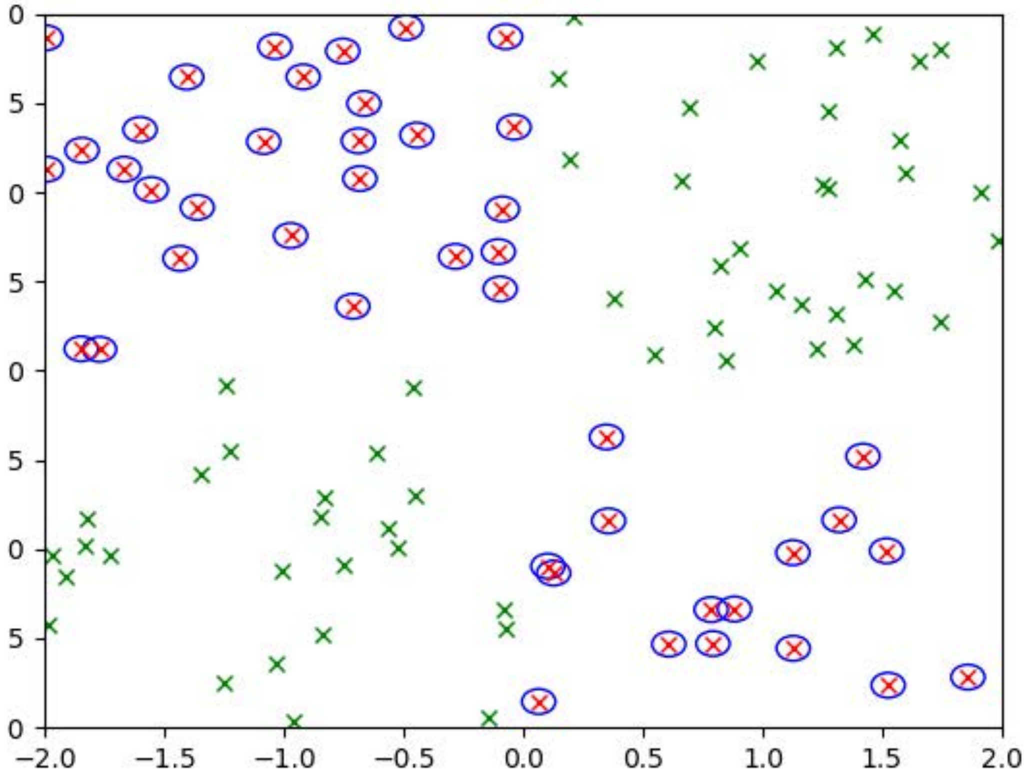


—> Use a validation set.

Failure Mode



Training (100 iterations)



Validation (56% accuracy)

- Individual linear classifiers cannot do better than chance!
- Box classifiers would work though.

Adaboost in Python

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def fit(self,nit=10):
    # Initialize weights and list of classifiers
    self.weakCls  = []
    bestAcc = 0.0
    self.datCoeffs = np.ones(self.ns,dtype=np.float)/self.ns
    # Find nit weak classifiers and update weights each time.
    for m in range(nit):
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        self.weakCls.append(weakC)
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```

```
def updateWeights(self,weakC):
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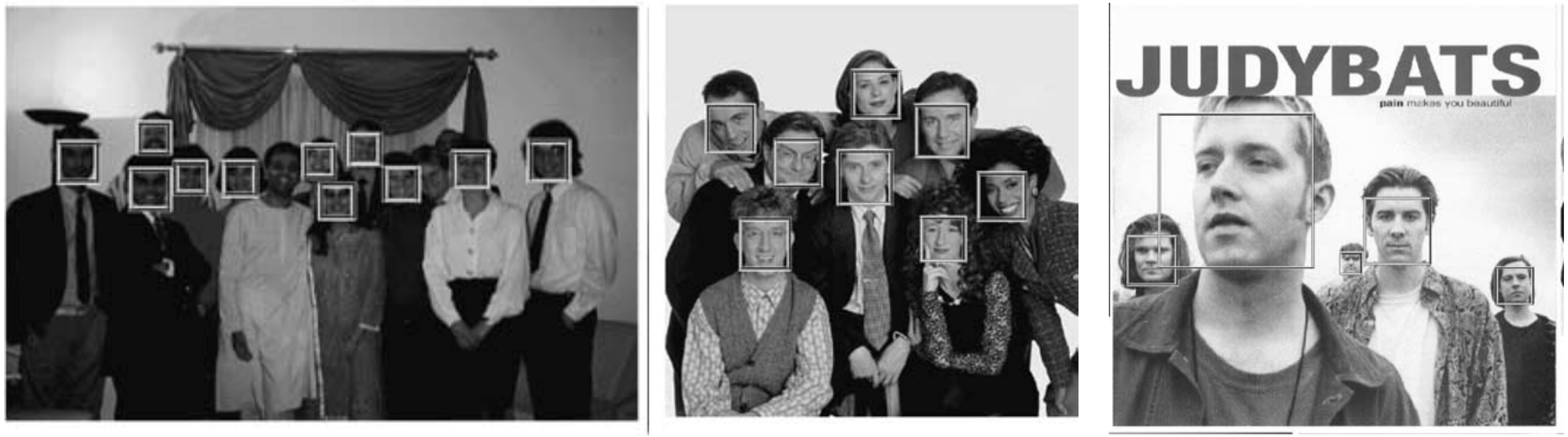
    err,_ = self.weakClassError(weakC)
    alpha = np.log(1.0/max(1e-10,err)-1.0)
    # Compute numbers of misclassified samples.
    nerrs = np.logical_not(weakC.predict(self.xs)==self.ys)
    # Update and normalize weights.
    self.datCoeffs *=  np.exp(alpha*nerrs)
    self.datCoeffs /=  sum (self.datCoeffs)
    return alpha
```

- A strikingly simple algorithm that works well.
- The weak classifiers do not have to be linear classifiers.

—>Versatile and generic.

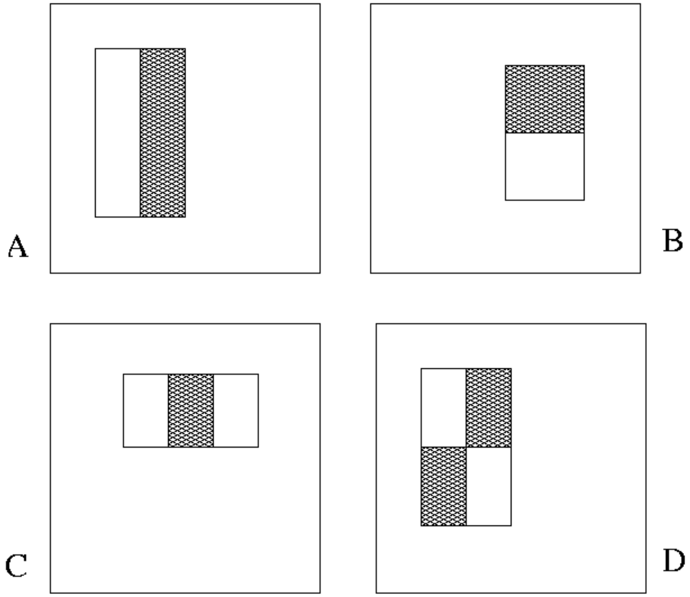
Face Detection

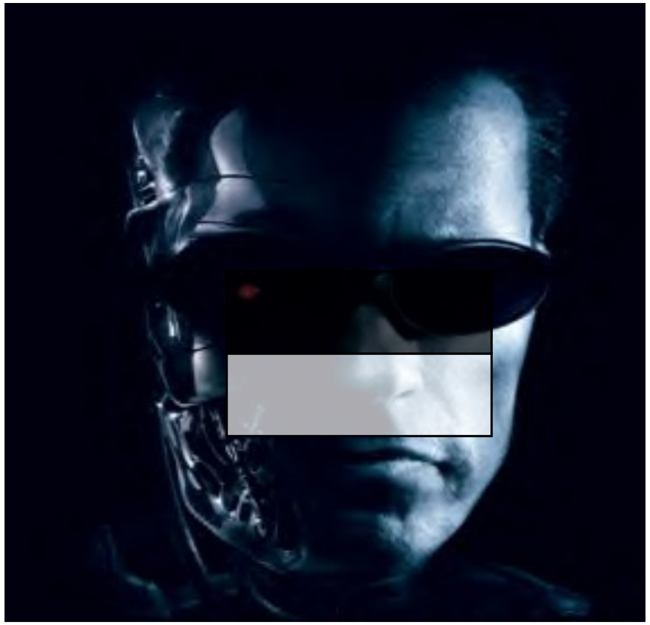


Viola & Jones, Rapid Object Detection using a Boosted Cascade of Simple Features, CVPR 2001:

- First reliable, real-time face detection system.
- Used in commercial products, such as digital cameras.

Weak Learners for Images





Value = ∑ (pixels in white area) – ∑ (pixels in black area)

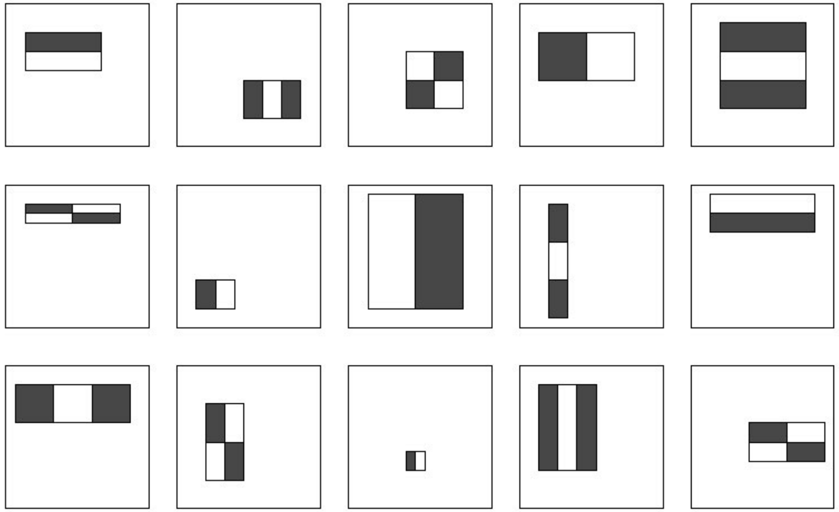
Haar Wavelets:

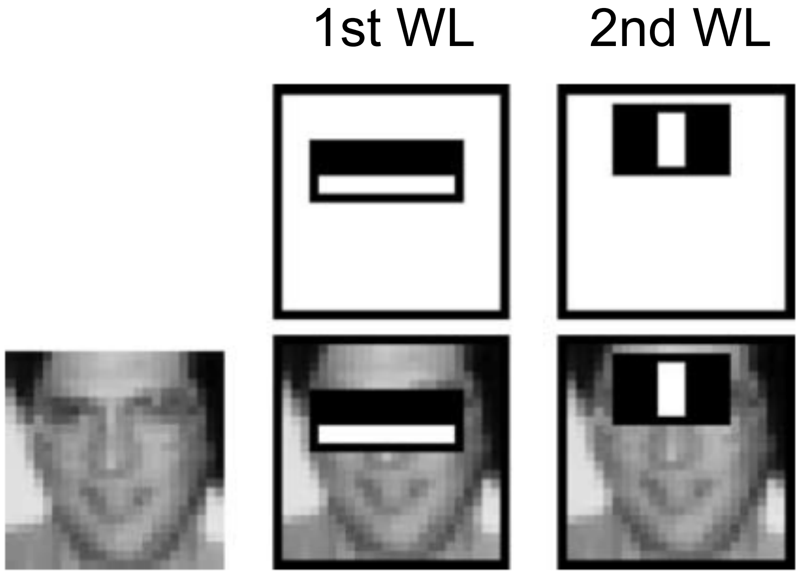
- Allow target function over an interval to be represented in terms of an orthonormal basis.
- Fast to compute (4 operations per rectangle).
- 180’000 possibilities for a 24x24 window.

—> Use AdaBoost to choose a good subset.

Feature Selection

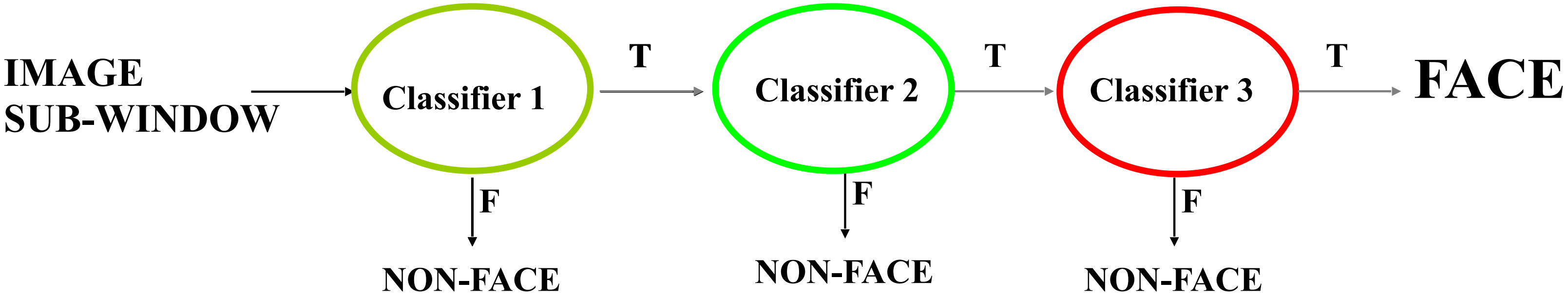
Among:





Pick:

Cascade



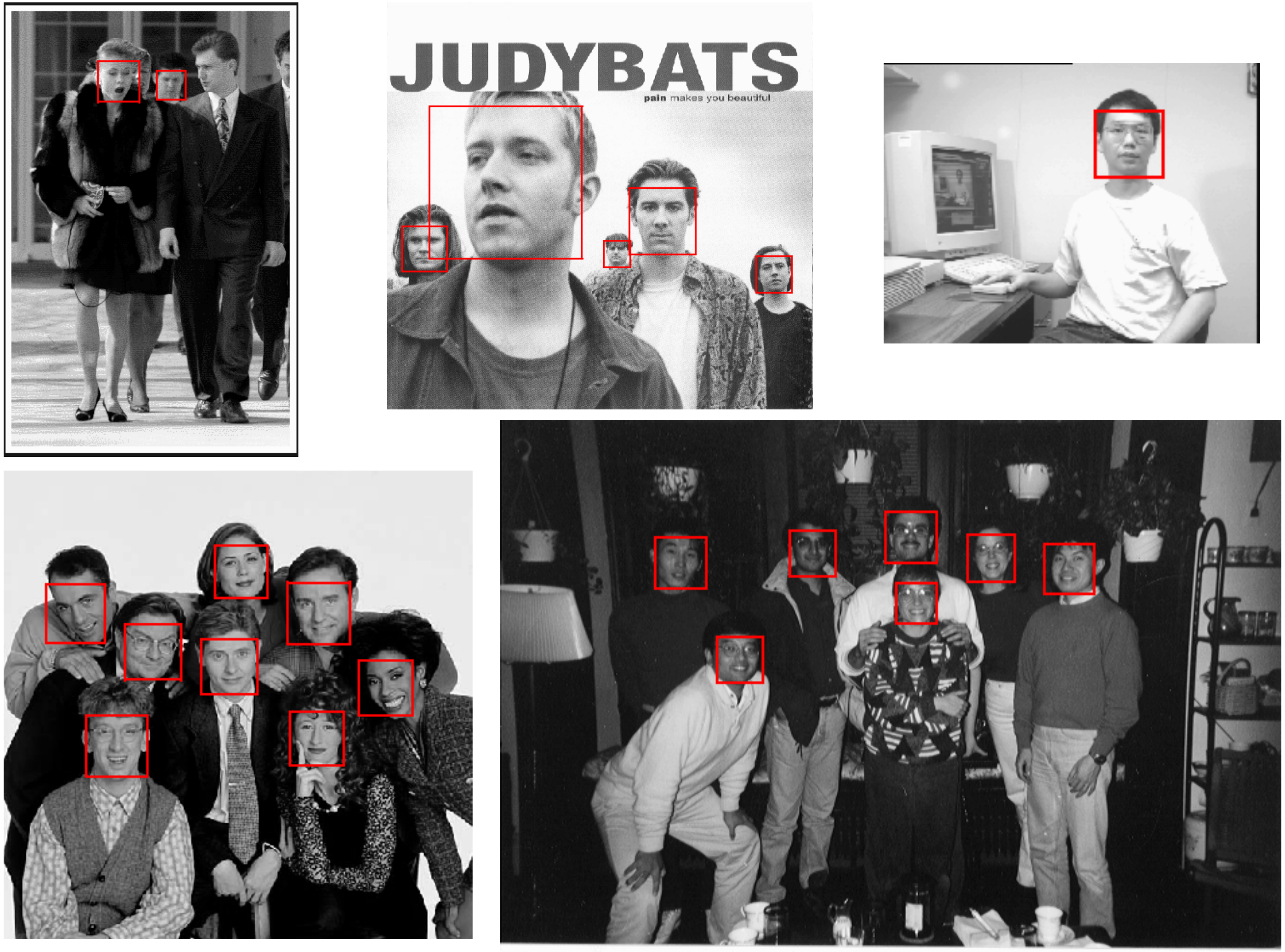
Reject large portions of the images using only the response of the first few weak classifiers.

—> Large potential speed-up at run-time.

Training Set

- Training Set
  - 5000 faces
    - All frontal, rescaled to 24x24 pixels
  - 300 million non-faces
    - 9500 non-face images
  - Faces are normalized
    - Scale, translation
- Many variations
  - Across individuals
  - Illumination
  - Pose

Detection Results (2001)



Detection Results (2017)