One Hundred Exercises: Solutions of the last five exercises

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June 24, 2021

Exercise 15.1 a) $S_n = \sum_{j=1}^n \xi_j$, so $X_n = \sum_{j=1}^n (n+1-j)\xi_j$ and

$$\mathbb{E}(X_n) = 0 \quad \text{and} \quad \mathrm{Var}(X_n) = \mathbb{E}(X_n^2) = \sum_{j=1}^n (n+1-j)^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$$

b)

$$\begin{aligned}\mathbb{E}(\exp(sX_n)) &= \prod_{j=1}^n \mathbb{E}(\exp(s(n+1-j)\xi_j)) = \prod_{j=1}^n \cosh(s(n+1-j)) \leq \prod_{j=1}^n \exp\left(\frac{s^2(n+1-j)^2}{2}\right) \\ &= \exp\left(\frac{s^2}{2}\sum_{i=1}^n (n+1-j)^2\right) = \exp\left(\frac{s^2\,\mathrm{Var}(X_n)}{2}\right).\end{aligned}$$

c) By Chebychev's inequality,

$$\mathbb{P}(X_n \geq n^2 t) \leq \frac{\mathbb{E}(\exp(sX_n))}{\exp(sn^2t)} \leq \exp\left(\frac{\mathrm{Var}(X_n)}{2}s^2 - n^2ts\right)$$

As $s > 0$ is a free parameter, we choose it so as to minimize $\left(\frac{\mathrm{Var}(X_n)}{2}s^2 - n^2ts\right)$, namely we choose $s^* = \frac{n^2t}{\mathrm{Var}(X_n)}$. As such, we get

$$\mathbb{P}(X_n \geq n^2t) \leq \exp\left(-\frac{1}{2}\frac{n^4t^2}{\mathrm{Var}(X_n)}\right)$$

and similarly

$$\mathbb{P}(X_n \leq -n^2t) = \mathbb{P}(-X_n \geq n^2t) \leq \exp\left(-\frac{1}{2}\frac{n^4t^2}{\mathrm{Var}(X_n)}\right)$$

d) For every $t > 0$, we have $\mathbb{P}(|Y_n| \geq t) = \mathbb{P}(|X_n| \geq n^2t) \leq 2\exp\left(-\frac{1}{2}\frac{n^4t^2}{\mathrm{Var}(X_n)}\right) \overset{n\to\infty}{\to} 0$, since $\mathrm{Var}(X_n) = \Theta(n^3)$. Also, $\sum_{n\geq 1}\mathbb{P}(|Y_n| \geq t) < \infty$, so by the Borel-Cantelli lemma, Y_n converges almost surely to zero.

Exercise 15.2 a) The random variables X_j are i.i.d. and bounded, so the same holds for $Y_j = (X_j-\mu)^3$, and therefore, the strong law of large numbers applies:

$$\frac{1}{n}\sum_{j=1}^n (X_j-\mu)^3 \underset{n\to\infty}{\to} \mathbb{E}((X_1-\mu)^3)(=\nu) \quad \text{almost surely}$$

b) No. Consider e.g. the case where $\mathbb{P}(\{X_1 = a\}) = p$ and $\mathbb{P}(\{X_1 = b\}) = 1-p$, with $0 < p < 1$. Then

$$\begin{aligned}\nu &= p\,(a-\mu)^3 + (1-p)\,(b-\mu)^3 = p\,(a-p\,a-(1-p)\,b)^3 + (1-p)\,(b-p\,a-(1-p)\,b)^3 \\ &= p\,(1-p)^3\,(a-b)^2 + (1-p)\,p^3\,(b-a)^3 = (-(1-p)^2+p^2)\,p\,(1-p)\,(b-a)^3 \\ &= (2p-1)\,p\,(1-p)\,(b-a)^3\end{aligned}$$

which is negative if $p < \frac{1}{2}$. [NB: Jensen's does not apply here, as $x \mapsto x^3$ is not convex]

c)

$$\mu = \frac{a+b}{2} \quad \text{and} \quad \nu = \frac{1}{2}(((a-b)/2)^3 - ((b-a)/2)^3) = 0$$

d) The random variables $(X_j - \mu)$ are i.i.d. and square-integrable, so the central limit theorem applies: $Z_n \underset{n\to\infty}{\overset{d}{\to}} Z \sim \mathcal{N}(0, \sigma^2)$, where

$$\sigma^2 = \mathrm{Var}(X_1) = \mathbb{E}((X_1-\mu)^2) = \frac{1}{2}\left(((a-b)/2)^2 + ((b-a)/2)^2\right) = \frac{(a-b)^2}{4}$$

e)

$$\mathbb{E}(Z_n^3) = \frac{1}{n^{3/2}} \sum_{i,j,k=1}^{n} \mathbb{E}((X_i-\mu)\,(X_j-\mu)\,(X_k-\mu))$$

As $\mathbb{E}((X_i-\mu)\,(X_j-\mu)\,(X_k-\mu)) = 0$ as soon as i, j, k are not all equal (by independence and the fact that $\mathbb{E}(X_j - \mu) = 0$ for all j), we obtain

$$\mathbb{E}(Z_n^3) = \frac{1}{n^{3/2}} \sum_{j=1}^{n} \mathbb{E}((X_j-\mu)^3) = \frac{1}{\sqrt{n}}\,\mathbb{E}((X_1-\mu)^3) = \frac{\nu}{\sqrt{n}}$$

Notice that this expression converges to 0 as $n \to \infty$, which is coherent with the fact seen above that $Z_n \underset{n\to\infty}{\overset{d}{\to}} Z \sim \mathcal{N}(0, \sigma^2)$.

Exercise 15.3 a) First remark: as $\exp(sXY) \geq 0$, $\mathbb{E}(\exp(sXY))$ is always well-defined, but possibly equal to $+\infty$. Now let us compute, using the hints:

$$\mathbb{E}(\exp(sXY)) = \mathbb{E}(\mathbb{E}(\exp(sXY)|X)) = \mathbb{E}(\exp(s^2X^2/2)) = \begin{cases} \frac{1}{\sqrt{1-s^2}} & \text{when } 0 \leq s < 1 \\ +\infty & \text{when } s > 1 \end{cases}$$

b) By the classical procedure, we have for every $0 \leq s < 1$:

$$\mathbb{P}(\{Z_n > nt\}) \leq e^{-snt}\,\mathbb{E}(\exp(sZ_n)) = e^{-snt} \prod_{j=1}^{n} \mathbb{E}(\exp(sX_jY_j)) = e^{-snt}\,\mathbb{E}(\exp(sX_1Y_1))^n$$
$$= \exp(-snt + n\log(1/\sqrt{1-s^2})) = \exp(-n(st + \frac{1}{2}\log(1-s^2)))$$

Therefore,

$$\mathbb{P}(\{Z_n > nt\}) \leq \inf_{0\leq s<1} \exp(-n(st + \frac{1}{2}\log(1-s^2))) = \exp\left(-n \sup_{0\leq s<1}(st + \frac{1}{2}\log(1-s^2))\right)$$

To show that the above supremum is greater than 0, observe that for any $t > 0$, $f(s) = st + \frac{1}{2}\log(1-s^2)$ satisfies

$$f(0) = 0 \quad \text{and} \quad f'(0) = t > 0$$

so there always exists $s > 0$ and $c = f(s) > 0$ such that $\mathbb{P}(\{Z_n > nt\}) \leq \exp(-cn)$.

Exercise 15.4 a) $S_0 = 0$, $S_1 = \pm 1$ wp 1/2, $S_2 = 0$, $S_3 = \pm 1$ wp 1/2, $S_4 = \begin{cases} +2 & \text{wp } 1/6 \\ 0 & \text{wp } 2/3 \\ -2 & \text{wp } 1/6 \end{cases}$.

b) No:

$$\mathbb{E}(S_{n+1}|\mathcal{F}_n) = S_n + \mathbb{E}(X_{n+1}|\mathcal{F}_n) = S_n + \left(\frac{1}{2} - \frac{S_n}{2n}\right) - \left(\frac{1}{2} + \frac{S_n}{2n}\right) = S_n - \frac{S_n}{n} = \frac{n-1}{n}\, S_n$$

but the process M defined as $M_n = (n-1)\, S_n$ is a martingale, as the following shows:

$$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = n\, \mathbb{E}(S_{n+1}|\mathcal{F}_n) = (n-1)\, S_n = M_n$$

c) First note that by the above computation, $\mathbb{E}(S_{n+1}) = \frac{n-1}{n}\, \mathbb{E}(S_n)$, and as $S_0 = 0$, this implies that $\mathbb{E}(S_n) = 0$ for all n. Let us then compute

$$\mathbb{E}(S_{n+1}^2|\mathcal{F}_n) = S_n^2 + 2S_n\, \mathbb{E}(X_{n+1}|\mathcal{F}_n) + \mathbb{E}(X_{n+1}^2|\mathcal{F}_n) = S_n^2 + 2S_n \left(-\frac{S_n}{n}\right) + 1 = \frac{n-2}{n}\, S_n^2 + 1$$

so

$$\mathrm{Var}(S_{n+1}) = \mathbb{E}(S_{n+1}^2) = \frac{n-2}{n}\, \mathbb{E}(S_n^2) + 1 = \frac{n-2}{n}\, \mathrm{Var}(S_n) + 1$$

Looking at the first terms of this recursion (or using any other analysis), one finds that $\mathrm{Var}(S_0) = 0$, $\mathrm{Var}(S_1) = 1$, $\mathrm{Var}(S_2) = 0$, $\mathrm{Var}(S_3) = 1$, $\mathrm{Var}(S_4) = 4/3$ (in accordance with what was found in part a), and then $\mathrm{Var}(S_n) = \frac{n}{3}$ for $n \geq 4$; this can be checked directly with the above formula.

A simpler result can be obtained by observing that $\mathrm{Var}(S_{n+1}) \leq \mathrm{Var}(S_n) + 1$, for all $n \geq 0$, so $\mathrm{Var}(S_n) \leq n$ for all $n \geq 0$.

d) Since $\mathrm{Var}(S_n) = O(n)$ (cf. part c), we conclude by Chebyshev's inequality that for any fixed $\varepsilon > 0$,

$$\mathbb{P}(\{S_n/n \geq \varepsilon\}) \leq \frac{\mathbb{E}(S_n^2)}{n^2 \varepsilon^2} = \frac{\mathrm{Var}(S_n)}{n^2 \varepsilon^2} = O\left(\frac{1}{n}\right)$$

implying convergence in probability towards 0.

Exercise 15.5 a) By Jensen's inequality, $\log \mathbb{E}(\exp(M_{n+1} - M_n)|\mathcal{F}_n) \geq \mathbb{E}(M_{n+1} - M_n|\mathcal{F}_n) = 0$, so the process A is increasing. By induction, we see that if A_n is $\mathcal{F}_{n-1}$-measurable, then A_{n+1} is $\mathcal{F}_n$-measurable, as $\log \mathbb{E}(\exp(M_{n+1} - M_n)|\mathcal{F}_n)$ is $\mathcal{F}_n$-measurable by definition.

b) Observe first that $\mathbb{E}(|X_n|) = \mathbb{E}(X_n) \leq \mathbb{E}(\exp(M_n)) \leq \exp(1)$, as $A_n \geq 0$ and $M_n \leq 1$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \mathbb{E}(X_{n+1}|\mathcal{F}_n) &= \mathbb{E}(\exp(M_{n+1})|\mathcal{F}_n) \exp(-A_{n+1}) = \mathbb{E}(\exp(M_{n+1})|\mathcal{F}_n)\, \exp(-A_n)\, \frac{1}{\mathbb{E}(\exp(M_{n+1} - M_n)|\mathcal{F}_n)} \\ &= \exp(M_n - A_n) = X_n \end{aligned}$$

Let now M be the martingale defined recursively as $M_0 = x \in]0, 1[$, $M_{n+1} = \begin{cases} M_n^2, & \text{with prob. } \frac{1}{2} \\ 2M_n - M_n^2, & \text{with prob. } \frac{1}{2} \end{cases}$

and A, X be the processes defined above in this particular case.

c) We have

$$A_{n+1} - A_n = \log\left(\frac{1}{2} \exp(M_n^2 - M_n) + \frac{1}{2} \exp(2M_n - M_n^2 - M_n)\right) = \log(\cosh(M_n(1 - M_n)))$$

so $A_n = \sum_{j=0}^{n-1} \log(\cosh(M_j(1 - M_j)))$.

d) We have seen above that $0 \leq X_n \leq \exp(1)$ for all $n \in \mathbb{N}$, so X is a bounded martingale and the first version of the martingale convergence theorem applies. So yes, there exists a random variable X_∞ such that $\mathbb{E}(X_\infty|\mathcal{F}_n) = X_n$ for all $n \in \mathbb{N}$.