Reflection principle

(leading to the computation of $f_{00}(2n)$ for the symmetric random walk)

Let us start with a preliminary computation:

Markov

$$f_{00}(2n) = \mathbb{P}(X_{2n}=0, X_{2n-1}\neq 0, \ldots, X_1 \neq 0 \mid X_0 = 0)$$

$$= \mathbb{P}(X_{2n}=0, X_{2n-1}>0, \ldots, X_2>0 \mid X_1=+1, \cancel{X_0=0}) \cdot \underbrace{\mathbb{P}(X_1=+1 \mid X_0=0)}_{=1/2}$$

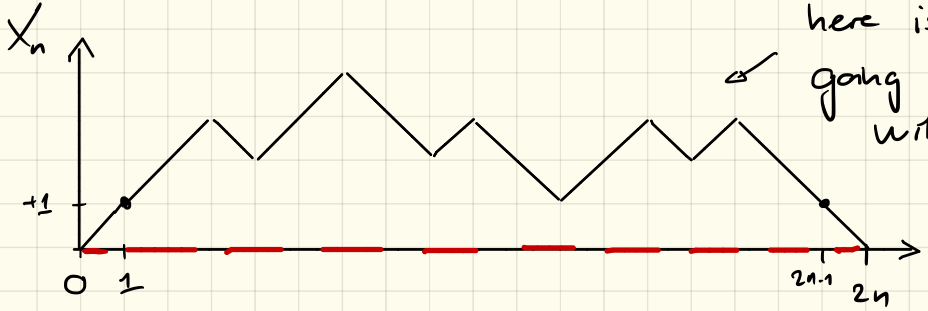
$$+\mathbb{P}(X_{2n}=0, X_{2n-1}<0, \ldots, X_2<0 \mid X_1=-1, \cancel{X_0=0}) \cdot \underbrace{\mathbb{P}(X_1=-1 \mid X_0=0)}_{=1/2}$$

$$= \mathbb{P}(X_{2n}=0, X_{2n-1}>0, \ldots, X_2>0 \mid X_1=+1) \quad \text{by symmetry}$$

$$= \mathbb{P}(X_{2n}=0, X_{2n-1}=+1, X_{2n-2}>0, \ldots, X_2>0 \mid X_1=+1)$$

$$= \underbrace{\mathbb{P}(X_{2n}=0 \mid X_{2n-1}=+1)}_{=1/2} \cdot \mathbb{P}(X_{2n-1}=+1, X_{2n-2}>0, \ldots, X_2>0 \mid X_1=+1)$$

(the Markov property is also used here)



here is a typical path going from $X_1 = +1$ to $X_{2n-1} = +1$ without touching the red axis

First observation:

1. The total number of paths from $X_1 = +1$ to $X_{2n-1} = +1$

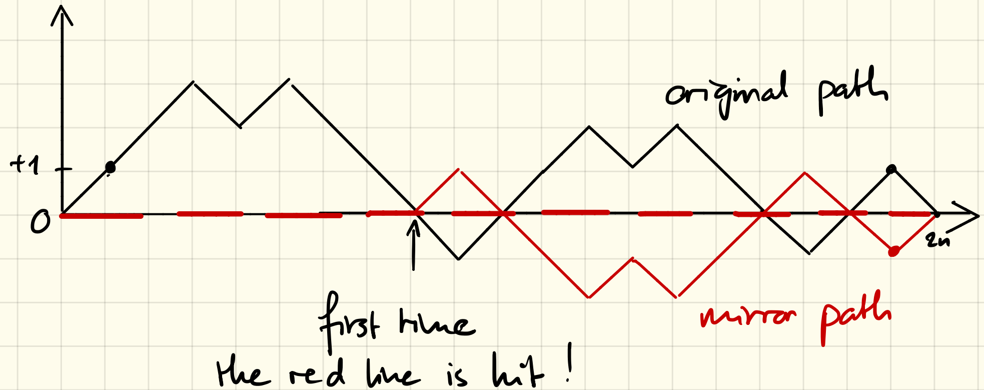
= the number of paths from $X_1 = +1$ to $X_{2n-1} = +1$ not touching the red line

+ the number of paths from $X_1 = +1$ to $X_{2n-1} = +1$ touching the red line

(& the probability of each path $= \frac{1}{2}^{2n-2}$
as $2n-2$ steps are performed from time 1 to time $2n-1$)

Second and third observations:

2. To each path touching the red line corresponds a "mirror path" defined as follows:



3. Every path going from $X_1 = +1$ to $X_{2n-1} = -1$ must touch the red line at least once.

Consequences:

- The number of paths from $X_1 = +1$ to $X_{2n-1} = +1$ touching the red line

  = the number of mirror paths from $X_1 = +1$ to $X_{2n-1} = -1$ touching the red line

  = the total number of paths from $X_1 = +1$ to $X_{2n-1} = -1$

- So: the number of paths from $X_1 = +1$ to $X_{2n-1} = +1$ not touching the red line

  = the total number of paths from $X_1 = +1$ to $X_{2n-1} = +1$

  − the total number of paths from $X_1 = +1$ to $X_{2n-1} = -1$

Therefore:

$$f_{00}(2n) = \frac{1}{2} \cdot \mathbb{P}(X_{2n-1} = +1, X_{2n-2} > 0, \dots, X_2 > 0 \mid X_1 = +1)$$

$$= \frac{1}{2} \cdot \left(\mathbb{P}(X_{2n-1} = +1 \mid X_1 = +1) - \mathbb{P}(X_{2n-1} = -1 \mid X_1 = +1) \right)$$

Now it is just a matter of counting the number of paths:

$$f_{00}(2n) = \frac{1}{2} \cdot \left(\binom{2n-2}{n-1} \cdot \frac{1}{2^{2n-2}} - \binom{2n-2}{n} \frac{1}{2^{2n-2}} \right)$$

n-1 times up & down

n times down, n-2 times up

$$= \frac{1}{2^{2n-1}} \left(\frac{(2n-2)!}{(n-1)!(n-1)!} - \frac{(2n-2)!}{n!(n-2)!} \right)$$

$$= \frac{1}{2^{2n-1}} \cdot \frac{(2n-2)!}{(n-1)!(n-1)!} \left(1 - \frac{n-1}{n} \right) = \frac{1}{2^{2n-1}} \cdot \frac{(2n-2)!}{(n-1)!\, n!}$$

Using then Stirling's approximations, one finds:

$$f_{00}(2n) \simeq \frac{1}{2n\sqrt{\pi(n-1)}} \quad \text{for large values of } n$$

So $\mathbb{E}(T_0 \mid X_0 = 0) = \sum_{n \geq 1} 2n \cdot f_{00}(2n)$

$$\sim \sum_{n \geq 1} \frac{1}{\sqrt{\pi(n-1)}} = +\infty$$

i.e. the symmetric random walk is null-recurrent.

QED