Lecture 4

last time

- We defined $T_pM = \{ v : C^\infty(M) \to \mathbb{R} \mid v$ is a derivation at $p \}$

And showed

- $\forall \; p \in U_p \subseteq M$ open nbd

$$T_p U \xrightarrow[D_p i]{\cong} T_p M$$

- For (U, φ) a smooth chart

$$D_p \varphi : T_pM \xrightarrow{\cong} T_{\varphi(p)} \hat{U} \cong T_{\varphi(p)} \mathbb{R}^n \cong \mathbb{R}^n_{\varphi(p)}$$

$\Rightarrow$ We can define a basis for T_pM as

$$\boxed{T_pM = \left\langle \frac{\partial}{\partial x^i}\Big|_p := D_{\varphi(p)} \varphi^{-1} \left(\frac{\partial}{\partial x^i}\Big|_{\varphi(p)} \right) \right\rangle_{i=1,\dots,n}}$$

Exercise: verify that you are happy with the following computations in local coordinates:

1. Let $f \in C^\infty(U)$, $p \in U$, and let $v = \sum v^i \frac{\partial}{\partial x^i}\Big|_p$

$$\Rightarrow \boxed{v(f) = \sum v^i \frac{\partial}{\partial x^i}\Big|_p (f) = \sum v^i \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} (\varphi(p))}$$

2. Let (U, φ), (V, ψ) local charts around $p \in M$, and $F(p) \in N$ respectively

with $\varphi(U) \subseteq \mathbb{R}^n_{x^1,\dots x^n}$

$\psi(V) \subseteq \mathbb{R}^m_{y^1,\dots y^m}$

$\Rightarrow$ $D_pF : T_pM \to T_pN$ in the basis

$\langle \frac{\partial}{\partial x^i}\big|_p \rangle$, $\langle \frac{\partial}{\partial y^i}\big|_{F(p)} \rangle$ is given by

$$\boxed{DF_p \left(\frac{\partial}{\partial x^i}\Big|_p\right) = \sum_{j=1}^{n} \frac{\partial \hat{F}^j}{\partial x^i} (\varphi(p)) \frac{\partial}{\partial y^j}\Big|_{F(p)}}$$

where $\hat{F} = \psi \circ F \circ \varphi^{-1}$

3. Change of coordinates:

let (U, φ), (U', φ') smooth coordinate charts around p

$$\varphi' \circ \varphi^{-1}(x^1, \dots x^n) = (z^1(\underline{x}), \dots, z^n(\underline{x}))$$

$\Rightarrow$ let $v \in T_pM$ s.t.

$$v = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^{n} Z^i \frac{\partial}{\partial z^i}\Big|_p$$

$$\Rightarrow \boxed{Z^i = \sum_{j=1}^{n} \frac{\partial z^i}{\partial x^j} X^j}$$

§ 4.1: Tangent Bundle

Definition: We call tangent bundle of M the disjoint union of all tangent spaces

$$TM = \bigsqcup_{p \in M} T_pM$$

This comes with a natural map

$$\pi : \begin{array}{ccc} TM & \to & M \\ \cup & & \\ v \in T_pM & \longmapsto & p \end{array}$$

Example

$$T\mathbb{R}^n = \bigsqcup_{e \in \mathbb{R}^n} T_e\mathbb{R}^n = \bigsqcup_{e \in \mathbb{R}^n} \{e\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \quad \text{and}$$

$\pi = p_1$ first projection.

Proposition (Lee's 3.18; 4.1.1 Notes)

Let M a smooth mnf of dim n

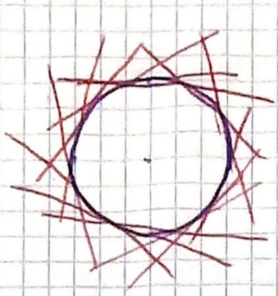
$\Rightarrow$ 1. $\exists$ a natural smooth structure on TM s.t. TM is a $2n$-dimensional mfd

2. $\pi : TM \to M$ is smooth with respect to these structure

3. Moreover, for any $F : M \to N$ smooth map

$$DF = \bigsqcup_p D_pF : TM \to TN \text{ is}$$

smooth



the vector spaces T_pM "fit" together smoothly

bundle

To prove the proposition we need the following

Lemma (smooth chart lemma; Notes 1.2.8; Lee's 1.35)

⌈ so far, we always presented a smooth manifolds by saying : - I start with a topological manifold; - choose an atlas i.e. collection of charts covering M s.t. transition functions are smooth ⌋

These lemma tells us that we can be more efficient some times ⌋

let M be a set. suppose ∃ a collection $\{U_\alpha\}$ of subsets in M s.t:

1) $\forall \alpha$ there is a bijection $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ open

2) $\forall \alpha, \beta$ $\varphi_\alpha(U_\alpha \cap U_\beta)$, $\varphi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$ are open

3) $\varphi_\beta^{-1} \circ \varphi_\alpha$ is smooth $\forall \alpha, \beta$

4) countably many $\{U_\alpha\}$ cover M

5) $\forall p, q \in M$ either $\exists U_\alpha \ni p, q$ or $\exists U_\alpha, U_\beta$ s.t. $U_\alpha \cap U_\beta \neq \emptyset$ and $p \in U_\alpha$, $q \in U_\beta$

$\Rightarrow \exists!$ smooth manifold structure on M s.t. $(U_\alpha, \varphi_\alpha)$ are smooth charts.

<u>proof</u>

• We define a topology on M by taking all the sets $\varphi_\alpha^{-1}(V)$ to be open $\forall\, V \underset{\text{open}}{\subseteq} \mathbb{R}^n$

• This is a base for a topology:
to show it, we need to show that given

$$p \in \varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$$

there exist a open $\varphi_\alpha^{-1}(U)$ s.t.

$$p \in \varphi_\alpha^{-1}(U) \subseteq \varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$$

Now notice that

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}\Big(\underbrace{V \cap (\varphi_\beta \circ \varphi_\alpha^{-1})^{-1}(W)}_{\text{this is a open in } \mathbb{R}^n}\Big) \quad \boxed{A}$$

• Each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a homeomorphism by def

(4) $\Rightarrow$ second countable

(5) $\Rightarrow$ Hausdorff

(3) $\Rightarrow \{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas $\boxed{E}$

proof of proposition

Ⓐ For each (U, φ) smooth chart for M

$$\pi^{-1}(U) \subseteq TM$$
$$\| $$
$$\bigsqcup_{p\in U} T_pU$$

$\Rightarrow$ If $(x^1, \_ x^n)$ are the coordinates on $\varphi(U)$ we can define

$$\tilde{\varphi} : \pi^{-1}(U) \longrightarrow \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$$
$$\Downarrow$$
$$v = \sum v^i \frac{\partial}{\partial x^i}\Big|_p \longmapsto (x^1(p), \_ x^n(p), v^1, \_ v^n)$$

$\tilde{\varphi}$ is a bijection onto a open of $\mathbb{R}^{2n}$

$$\Downarrow$$

$$\tilde{\varphi}^{-1}(x^1, \_ x^n, v^1, \_ v^n) := \sum v^i \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x)}$$

- We show that the collection of $(\pi^{-1}(U), \tilde{\varphi})$ satisfy the hp of the smooth chart lemma

(1) ✓ just did

(2) given (U, φ), (V, ψ) two smooth charts, consider $(\pi^{-1}(U), \tilde{\varphi})$, $(\pi^{-1}(V), \tilde{\psi})$

$\Rightarrow \quad \tilde{\varphi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \varphi(U \cap V) \times \mathbb{R}^n$

$\tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \psi(U \cap V) \times \mathbb{R}^n$

both open in $\mathbb{R}^{2n}$ since φ, ψ are smooth charts.

3) let us write down explicitly the transition function (Exercise)

$$\tilde{\psi} \circ \tilde{\varphi}^{-1}(x^1, \dots x^n; v^1, \dots v^n)$$

$$= \left(z^1(x), \dots z^n(x); \sum \frac{\partial z^1}{\partial x^j}(x) v^j, \dots \sum \frac{\partial z^n}{\partial x^j}(x) v^j \right)$$

Since $z^i(x)$ are smooth $\Rightarrow \tilde{\psi} \circ \tilde{\varphi}^{-1}$ is smooth

4) since M is a manifold, we can choose a countable cover $\{U_i\}_{i \in \mathbb{N}}$

$\Rightarrow \{ \pi^{-1}(U_i), \tilde{\varphi}_i \}_{i \in \mathbb{N}}$ is a countable cover

5) If $(p, v), (q, w) \in T_pM$ i.e. $p = q$

$\Rightarrow$ they are contained in the same chart ✓

If $p \neq q \Rightarrow \exists\ U_p\ U_q$ in M s.t.

$U_p \cap U_q = \emptyset \Rightarrow \pi^{-1}(U_p), \pi^{-1}(U_q)$ are disjoint.

∎

(3) to show that $\pi : TM \to M$ is smooth, choose local charts

$(U, \varphi), (\pi^{-1}(U), \tilde{\varphi})$

$\Rightarrow \quad \varphi \circ \pi \circ \tilde{\varphi}^{-1} : \varphi(U) \times \mathbb{R}^n \longrightarrow \varphi(U)$ is

just the first projection ✓ ∎

(c) Let $F : M \to N$ a smooth manifold

to show that

$dF : TM \longrightarrow TN$ is smooth

Let us look at the expression of dF

in local coordinates:

$$dF(x^1, \ldots x^n, v^1, \ldots v^n) = \left(\hat{F}^1(x^1), \ldots \hat{F}^n(x), \sum \frac{\partial \hat{F}^1}{\partial x^i}(x) v^i, \ldots \right)$$

In the next one, what we will do is start studying properties of maps between manifolds by studying certain properties of their differentials

↳ you already saw this philosophy in echin is vector calculus

§ 4.2: Maps of constant rank
(chapter 4 Lee) chapter 3 notes)

Definition, let $F : M \to N$ a smooth manifold.
We define the rank of F at p to be the rank of the linear map
$$dF_p : T_pM \to T_{F(p)}N$$

- F is said to have constant rank if rank of d_pF is the same at all p.
- F is called a smooth submersion if d_pF is surjective at every point, i.e $\boxed{\text{rank} = \dim N}$
- F is called a smooth immersion if d_pF is injective at every point, i.e $\boxed{\text{rank} = \dim M}$ → embedded submanifold
- F is an smooth embedding if it is a smooth immersion and $F : M \to F(M)$ is a homeomorphism, with F(M) having the subspace topology

↳ In this case we call F(M) an embedded submanifold of N

smooth submanifold of $\mathbb{R}^n$ are examples of embedded manifolds

Examples

① the standard immersion $i^n_K : \mathbb{R}^K \to \mathbb{R}^n$ $K \leq n$

$$i^n_K : (x^1, \ldots x^K) \mapsto (x^1, \ldots x^K, 0, \ldots 0)$$

is a smooth embedding (diffeomorphism onto its image)

② the standard submersion $\pi^K_m : \mathbb{R}^m \to \mathbb{R}^K$ $m \geq K$

is defined as

$$\pi^K_m (x^1, \ldots x^m) = (x^1, \ldots x^K)$$

③ the standard rank K map

$$i^n_K \circ \pi^K_m : \mathbb{R}^m \to \mathbb{R}^n$$

$$(x^1, \ldots x^m) \to (x^1, \ldots x^K, 0 \ldots 0)$$

④ given M_1, M_2 smooth manifold, $\exists$ a natural smooth manifold structure on $M_1 \times M_2$ with respect to which

$\pi_i : M_1 \times M_2 \to M_i$ is a smooth submersion

⑤ let $J \subseteq \mathbb{R}$ a open interval

And let $\gamma : J \to M$

a smooth curve.

$\Rightarrow$ γ is a smooth immersion

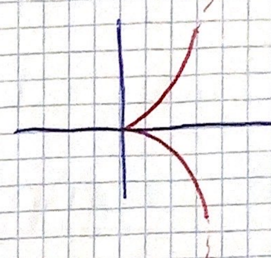
$\Leftrightarrow$ $\boxed{|\gamma'(t)| \neq 0 \quad \forall\, t \in J}$ $\quad d_p\gamma : \mathbb{R} \to T_{\gamma(p)}M$

Take $\gamma : J \to \mathbb{R}^2$ defined by

$\rightarrow$ $t \to (\cos t, \sin t)$ $t \in \mathbb{R}$ this is a smooth immersion

$t \in [0, 2\pi)$

$\rightarrow$ $t \to (t^2, t^3)$ is not a smooth immersion

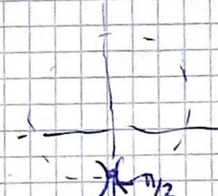


$\rightarrow$ The bent line

$C = \{(x,y) \in \mathbb{R}^2 \mid xy = 0,\ x,y \geq 0\}$ is not an embedded submanifold, i.e it is not the image of a smooth immersion

(Exercise)

$\rightarrow$ The figure 8:

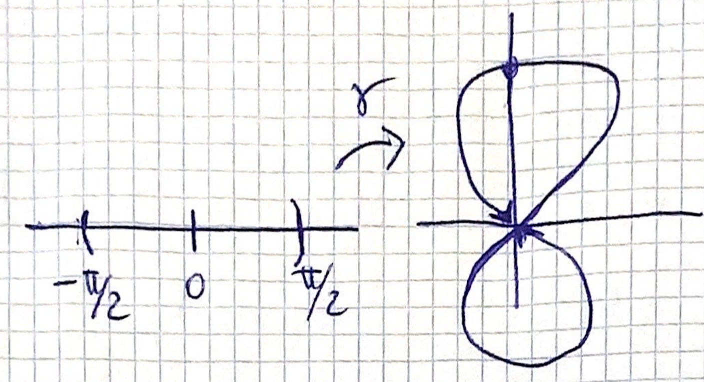


$\gamma : (-\frac{\pi}{2}, \frac{3\pi}{2}) \to \mathbb{R}^2$

$t \to (\sin(2t), \cos(t))$

- γ is a smooth immersion
- γ is inj suppose $\cos(t_1) = \cos(t_2) \Rightarrow t_2 = t_1 + \pi$

$\sin(2t_2) = \sin(2(t_1 + \pi)) = \ldots$



let take $\gamma(0) = (0,1)$

$\Rightarrow$ · γ is injective

· γ is a smooth immersion

· γ is not an embedding

$\gamma(J)$ is compact with the subspace topology

⑥ $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding

⑦ Graphs Let $F : M \to N$ a smooth map,

$$\Gamma(F) = \{ (x, F(x)) \in M \times N \} \subseteq M \times N$$

$$\Rightarrow \quad g : M \longrightarrow M \times N$$
$$x \longrightarrow (x, F(x))$$

is an embedding with image $\Gamma(F)$

→ smooth immersion

$$D_p g : T_pM \to T_{g(p)}(M \times N) \cong T_pM \times T_{F(p)}N$$

$$v \mapsto (v, D_pF(v))$$

⇒ $rk = \dim M$.

→ injective

→ embedding we have

$$\begin{tikzcd} M \times N \arrow[dr, "p_1"] & \\ \Gamma(F) \arrow[u, phantom, "\cup|"] \arrow[r] & M \end{tikzcd}$$

$$g^{-1} : \Gamma(F) \to M$$
$$(x, F(x)) \to x$$

(8) We can also have a smooth map which is a topological embedding but not a smooth embedding

$$\gamma : \mathbb{R} \to \mathbb{R}^2$$
$$t \to (t^3, 0)$$

⇒ $\gamma'(0) = 0$ ⇒ not an immersion

Proposition

Let $F : M \to N$ a smooth immersion s.t. F is also injective. Then F is a smooth embedding if one of the following holds

(a) F is open or closed.

(b) F is proper

(c) M is compact

(d) $\dim M = \dim N$

<u>proof</u>

$\ulcorner\!\!\rightarrow f$ inj by hp

(c) $f : M \to f(M)$ is bijective and continuos. If f is open or closed

$\Rightarrow f^{-1} : f(M) \to M$ is also continuos

$\Rightarrow f$ is a homeomorphism ✓

(d) If $\dim M = \dim N$

$\Rightarrow$ d_pF is an isomorphism

$\Rightarrow$ (Inverse Function Theorem) D_pF is a local diffeomorphism $\Rightarrow$ open

(c) A map from a compact space to a Hausdorff Space is closed. Indeed,

we want to show that $C \subseteq M$ closed

$f(C)$ is closed

- M cpct $\Rightarrow$ C compact
- $f(C)$ is also compact
- A ~~compact~~ subset in X Hausdorff is closed

$\ulcorner$ we will show that $X \setminus K$ is open

$\Leftrightarrow \forall x \in X \setminus K \ \exists \ U_x \subseteq X \setminus K$

open $\lrcorner$ ~~[illegible]~~

(b) F is proper $\Rightarrow F^{-1}(K)$ is compact if K compact.

We want to show that F is closed

~~[illegible]~~

Let $C \subseteq M$ closed

We want to show that $F(C)$ is closed

$\Leftrightarrow Y \setminus F(C)$ open.

$\forall\, y \in Y \setminus F(C)$ consider $\overline{B_r} \ni y$

$\Rightarrow F^{-1}(\overline{B}_r)$ is compact

consider $E = C \cap F^{-1}(\overline{B})$ this is compact

$\Rightarrow F(E)$ compact $\Rightarrow U = \overline{B} \setminus F(E)$ is open disj from $F(C)$ $\Rightarrow N \setminus F(C)$

(F) The image $F(M)$ has a open ngb W s.t. $F: M \to W$ is closed

Proposition (Existence of tubular neighborhood)

$F: M \to N$ smooth map. F is an embedding $\Leftrightarrow \exists$ a smooth retraction $g: U \to M$ for $F(M) \subseteq U \subseteq N$

i.e. $g \circ F = id_M$

Proof

$D_p\, id_M = D_{F(p)}\, g \circ D_p F \Rightarrow$
- D_pF inj $\forall p$ $\Rightarrow$ smooth immersion
- $g|_{F(M)} = F^{-1}$, continuous