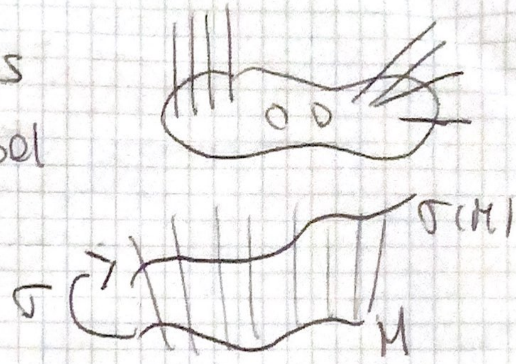
Lecture 8: vector fields & flows

Last time

- vector bundles
- local and global sections and frames.



$$\begin{array}{ccc} E & \hookleftarrow & E_p \\ \downarrow \pi & & \downarrow \\ M & \supset & p \end{array}$$

Today

$E = TM \xrightarrow{\pi} M$

- A section $X : M \to TM$ is called <u>vector field</u>
- $\mathfrak{X}(M)$ is the vector space of global vector fields

Why we care?

① If $F : \mathbb{R}^n \to \mathbb{R} \in C^\infty(\mathbb{R}^n)$ Then the partial derivatives $\frac{\partial F}{\partial x^i} := \frac{\partial}{\partial x^i}(F) \in C^\infty(\mathbb{R}^n)$

For now on a manifold M we know only how to define $\forall p, v \in T_pM$ $\quad v(F) = \sum_{i=1}^{n} \frac{\partial (F \circ \varphi^{-1})}{\partial x^i}(p) \cdot v^i$

$\rightsquigarrow$ <u>derivative at a point p</u>

② In $\mathbb{R}^n$, we can consider $\frac{\partial}{\partial x^j}\left(\frac{\partial F}{\partial x^i}\right) := \frac{\partial F}{\partial x^i \partial x^j} = \frac{\partial F}{\partial x^j \partial x^i}$

is this also true on M?

If not, what changes

③ How does dynamics, trajectories work on manifolds?

Examples of vector fields

(1) On $\mathbb{R}^n$, $\frac{\partial}{\partial x^i} : \mathbb{R}^n \longrightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$

$$p \longmapsto \Big(p, \underbrace{\frac{\partial}{\partial x^i}\Big|_p}\Big)$$

notation is the vector $e_i|_p$ thought as a derivation

(2) TM

$\downarrow$

$M \supset (U, \varphi)$ coordinate chart

$$\Rightarrow TM|_U \xrightarrow{\simeq} U \times \mathbb{R}^n$$

$$\sum_p v^i \frac{\partial}{\partial x^i}\Big|_p \longleftarrow\!\shortmid (p, v)$$

$\Rightarrow$ on U we have that every vector fields can be written as $V = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}$

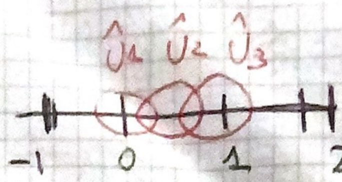
where $v^i : U \longrightarrow \mathbb{R}$ are smooth functions

$$V_p := V(p) = \sum_{i=1}^n V^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

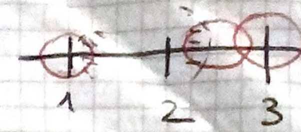
$$\frac{\partial}{\partial x^i}\Big|_p := (D_p\varphi)^{-1}(e_i|_p)$$

(3) On S^1 we have a nowhere vanishing vector field, which it is often denoted by $\frac{\partial}{\partial \theta}$

It is constructed this way!



Recall $S^1 = \mathbb{R}/\mathbb{Z} \xleftarrow{\kappa} \mathbb{R}$



$\Rightarrow$ we have charts $\varphi^{-1} : \hat{U} \subseteq \mathbb{R} \longrightarrow U$
open set

$\varphi^{-1} \cong \tilde{\varphi}$ over which κ is injective

On each of these charts we have

$$D\tilde{\varphi} : \hat{U} \times \mathbb{R} \longrightarrow TU \quad \text{is a diffeomorphism}$$
$$\left(\varphi(p), \frac{\partial}{\partial x}\right) \longmapsto \frac{\partial}{\partial x}\Big|_p$$

Claim:

$\exists$ a global vector field $\frac{\partial}{\partial \theta} : S^1 \to TS^1$ which on local charts is the one described above

proof

We have to show that given $(U, \varphi), (V, \psi)$ two charts if we consider

$$\begin{tikzcd} D(\psi \circ \varphi^{-1}) : T(\hat{U} \cap \hat{V}) \arrow[rr] \arrow[dr] & & T(\hat{U} \cap \hat{V}) \arrow[dl] \\ & \hat{U} \cap \hat{V} & \end{tikzcd}$$
$$D(\psi \circ \varphi^{-1}) = \mathrm{Id}$$

$\Rightarrow$ Implies that the vector field glue

The transition function

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$
$$\psi \circ \varphi^{-1} = t_n \qquad x \longmapsto x + n$$

$\Rightarrow Dt_n = \mathrm{Id} \Rightarrow$ we can define it globally!

(4) On $\mathbb{R}^2 \setminus \{0\}$, set $r(x,y) = \sqrt{x^2+y^2}$

$$\Rightarrow \quad E_1 = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y}$$

$$E_2 = -\frac{y}{r}\frac{\partial}{\partial x} + \frac{x}{r}\frac{\partial}{\partial y}$$

give $\forall p \in \mathbb{R}^2 \setminus 0$ an orthonormal base for $T_p(\mathbb{R}^2 \setminus \{0\})$

<u>Vector Fields as derivation</u>

<u>Definition</u>: An $\mathbb{R}$-linear map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if

$$D(f g) = f D(g) + g D(f)$$

given $X \in \mathfrak{X}(M)$ a vector field

$X(f) : M \longrightarrow \mathbb{R}$ is the function defined by

$$X(f)(p) := X_p f$$

choose a chart $(U, \varphi) \Rightarrow X|_U := \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$

$X^i \in C^\infty(U)$ and $\left.\frac{\partial}{\partial x^i}\right|_p = (D_p\varphi)^{-1}(e_i|_p)$

$$\Rightarrow \quad X(f)(p) = X_p f = \sum_{i=1}^{n} X^i(p) \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))$$

This shows that $X(f) \in C^\infty(M)$

The fact that $X(f \cdot g) = f X(g) + g X(f)$ we can check pointwise. But we already know that

$X_p \in T_pM$ has this property
(we saw it in lecture 3)

Vector fields and smooth maps

Recall that given $F: M \to N$ smooth we get

$$\begin{array}{rccc} DF: & TM & \longrightarrow & TN \\ & \downarrow & & \downarrow \\ F: & M & \longrightarrow & N \end{array}$$

In particular $\forall\, p \in M$ we have $D_pF: T_pM \to T_{F(p)}N$

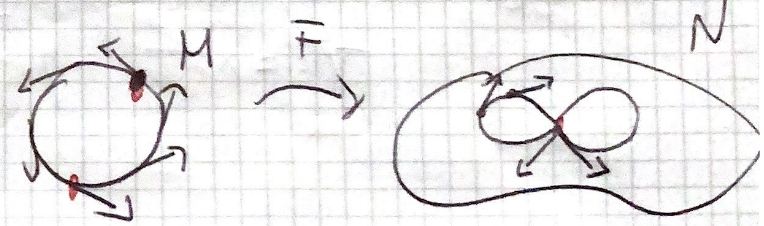
Let $X \left(\begin{array}{c} TM \\ \downarrow \pi \\ M \end{array} \right.$ be a smooth vector field

$\Rightarrow \forall\, p$ we can define $D_pF(X_p) \in T_{F(p)}N$

Attention though!!
We cannot define (in general) a vector field on N by setting $Y_p := D_pF(X_p)$

① If F is not surjective, at a point $q \in N \setminus F(M)$ we don't know how to define Y_q

② If F is not injective



$$\Rightarrow Y_{F(p_1)=F(p_2)} \overset{?}{=} d_{p_1}F(X_{p_1})$$
$$\overset{?}{=} d_{p_2}F(X_{p_2})$$

Definition: Let

$X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ we say that they are F-related if $\forall\ p \in M$

$$\boxed{dF_p(X_p) = Y_{F(p)}}$$

In general given a vector field $X \in \mathfrak{X}(M)$ it might not exist an F-related vector field Y

Exercise

If X, Y are F-related, then $\forall\ f \in C^\infty(N)$

$$X(f \circ F) = Y(f) \circ F \in C^\infty(M)$$

Example

$F: \mathbb{R} \to \mathbb{R}^2$ $\quad x(t) = \cos t$, $y(t) = \sin t$

$\Rightarrow \frac{\partial}{\partial t}$ is F Related to

$$Y := x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$DF : T\mathbb{R} \longrightarrow T\mathbb{R}^2$$

$$\mathbb{R}_t \times \langle \tfrac{\partial}{\partial t} \rangle \mathbb{R} \quad \begin{vmatrix} -\sin t \\ \cos t \end{vmatrix} \quad \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y} \rangle \mathbb{R} \times \mathbb{R}^2$$

$$Y_{F(t)} = \cos t \frac{\partial}{\partial y} - \sin t \frac{\partial}{\partial x}$$

$$D_t F\left(\frac{\partial}{\partial t}\right) = -\sin t \frac{\partial}{\partial x} + \cos t \frac{\partial}{\partial y}.$$

Proposition [Vector fields under change of base]

Let $F : M \to N$ a diffeomorphism

$\Rightarrow$ given $X \in \mathfrak{X}(M)$, $\exists!$ $Y \in \mathfrak{X}(M)$ s.t. Y and X are F-related.

Proof

Define $\boxed{Y_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)})}$

clearly this is the only ~~way~~ to define Y

why is it smooth?

$$\begin{tikzcd} N \arrow[r, "F^{-1}"] \arrow[rrr, bend right, "Y"'] & M \arrow[r, "X"] & TM \arrow[r, "dF"] & TN \end{tikzcd}$$

$\Rightarrow$ smooth.

We denote by $F_* X$ the vect field

$$(F_* X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)})$$

If you know how to compute F^{-1}
$\Rightarrow$ the formula give you an explicit description for $F_* X$

Lie Bracket

given $X, Y \in \mathfrak{X}(M)$ we define

$[X, Y]$ as the derivation

$$[X,Y] : C^\infty(M) \to C^\infty(M)$$

$$X(Y(f)) - Y(X(f))$$

- Verify that $[X,Y]$ is still a derivation

Example

Take $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial y}$ and $f(x,y) = x$, $g(x,y) = y$

$$\Rightarrow XY(f \cdot g) = 2x \neq f\, XY(g) + g\, XY(f) = x$$

In Fact:

Proposition Every derivation $D : C^\infty(M) \to C^\infty(M)$ is given by a vector field.

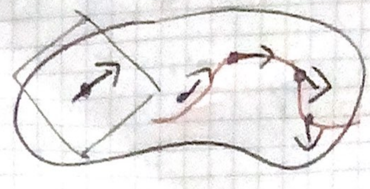
Proof

We have proved in lecture 3 that $D_p : C^\infty(M) \to \mathbb{R}$ is given by some $X_p \in T_pM$

$\Rightarrow$ Define X via the rule

$$X_p f = D_p f \quad \forall f \in C^\infty(M)$$

§ 8.2. Integral curve and Flows



Recall that given $v \in T_pM$

$\exists$ a curve

$\gamma : J \subseteq \mathbb{R} \to M$ s.t.

$\gamma(0) = p$ and $\boxed{v = \dot{\gamma}(t)}$ is the velocity vector

Definition: let $X \in \mathfrak{X}(M)$ a smooth vector field

$\gamma : J \subseteq \mathbb{R} \to M$ is an integral curve of X

if $\forall t \in J$

$$\boxed{X_{\gamma(t)} = \dot{\gamma}(t)}$$

$\gamma(0)$ is called the starting point of γ.

Example

① let $V = \frac{\partial}{\partial x}$ on $\mathbb{R}^2$, $V : \mathbb{R}^2 \to T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$

$(x,y) \mapsto \left((x,y), \frac{\partial}{\partial x}\Big|_{(x,y)}\right)$

$\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad e_1|_{(x,y)}$

$\gamma : J \subseteq \mathbb{R} \to \mathbb{R}^2$

$t \to (x(t), y(t))$

$\Rightarrow \dot{\gamma}(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y}$

$\|$

$D_t\gamma : T\mathbb{R} \to T\mathbb{R}^2$

$\Rightarrow$ we want $x'(t) = 1$, $y'(t) = 0$ $\Rightarrow$ $\gamma(t) = (a+t, b)$

If we choose the starting point $\gamma(0) = (x^0, y^0)$

$\Rightarrow$ a, b are fixed

$\Rightarrow \exists!$ integral curve for V starting at (x^0, y^0).

(2) $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$

$$\gamma(t) = (x(t), y(t))$$

$\Rightarrow$ $\dot{\gamma}(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y}$ is an int. curve for W

$$\Leftrightarrow \begin{cases} y'(t) = x(t) \\ x'(t) = -y(t) \end{cases} \Rightarrow \begin{aligned} x(t) &= a\cos t - b\sin t \\ y(t) &= a\sin t + b\cos t \end{aligned}$$

$\Rightarrow$ Find an Integral curve $\gamma : J \subseteq \mathbb{R} \to M$

we do as follows:

- choose coordinates chart (U, φ) for M
- on each chart U we can consider

$$\gamma : J \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$$
$$t \longmapsto (\gamma^1(t), \ldots, \gamma^n(t))$$

and $(X|_U) = \sum X^i(x^1, \ldots x^n)\frac{\partial}{\partial x^i}$

$\shortparallel$

$X|_U$

abuse of notation

- AN integral curve is the solution of the system of ordinary differential equations

$$\dot{\gamma}^1(t) = X^1(\gamma^1(t), \dots \gamma^n(t))$$

$$\vdots$$

$$\dot{\gamma}^n(t) = X^n(\gamma^1(t), \dots, \gamma^n(t))$$

Theorem (Fundamental Theorem of ODE)

Let $V : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ a smooth vect. valued funct.

Consider the initial value problem

$$(*) \quad \begin{aligned} &\dot{\gamma}^i(t) = V^i(\gamma^1(t), \dots \gamma^n(t)) \\ &\gamma^i(t_0) = c^i \quad , \; t_0 \in \mathbb{R}, \; (c_1, \dots c_n) \in U \end{aligned}$$

$\Rightarrow$

(1) $\forall$ $t_0 \in \mathbb{R}$, $c \in U$ $\exists$ $t_0 \in J_0 \subseteq \mathbb{R}$ and $c \in U_0 \subseteq U$ s.t.

$\exists$ $\quad \gamma : J_0 \to U$

s.t. γ solves $(*)$

(2) γ is unique

(3) let $\Theta : J_0 \times U_0 \to U$ be defined by

$(t, x) \mapsto \gamma(t)$ where $\gamma(t_0) = x$ is IC

$\Rightarrow$ Θ is smooth

Corollary: let V a vector field on M

$\forall$ $p \in M$ $\exists$ $J \subseteq \mathbb{R}$ and $\gamma : J \to M$ smooth with $\gamma(0) = p$ s.t. $V_{\gamma(t)} = \dot{\gamma}(t)$

Remark

- If γ is an integral curve for $V \in \mathfrak{X}(M)$

$$\Rightarrow \quad \tilde{\gamma}(t) := \gamma(at)$$

$\tilde{\gamma} : \tilde{J} = \{t \in \mathbb{R} \mid t \cdot a \in J\} \to M$ is an integral curve for $a \cdot V$

- If $\gamma : J \to \mathbb{R}$ is an integral curve for V

$$\Rightarrow \quad \hat{\gamma} : \hat{J} = \{t \in \mathbb{R} \mid t + b \in J\} \longrightarrow \mathbb{R}$$

$\hat{\gamma}(t) = \gamma(t+b)$ is also an int. curve for V, with starting point $\gamma(b)$

Definition: A global flow on M or (1-parameter group action) is a smooth map

$$\Theta : \mathbb{R} \times M \longrightarrow M$$

s.t

$$(1) \quad \Theta(0, p) = p$$

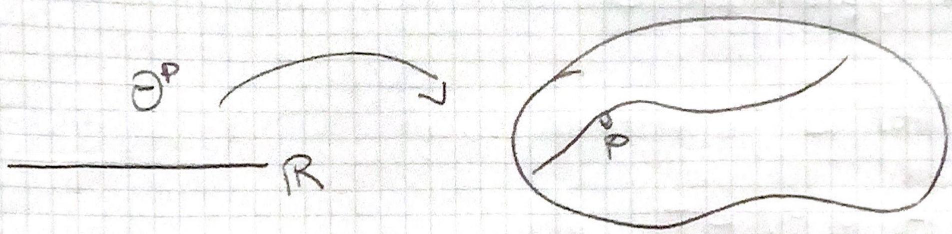
$$(2) \quad \Theta(t, \Theta(s, p)) = \Theta(t+s, p)$$

$\Leftrightarrow$ A global flow is the data of

- $\forall t \in \mathbb{R}$ $\Theta_t : M \to M$ a smooth map
- $\Theta_0 = id_M$
- $\Theta_t \circ \Theta_s = \Theta_{t+s}$

Given a global flow, $\forall p \in M$ we can define a curve

$$\boxed{\Theta^{(p)}(t) := \Theta(t, p)}$$



$\Rightarrow$ given Θ a smooth global flow we define a vector field $V \in \mathfrak{X}(M)$ by the rule

$$\boxed{V_p = \dot{\Theta}^{(p)}(0)}$$

V is in fact smooth = infinitesimal generator of Θ

$$\left[\ \forall f \in C^\infty(M) \quad Vf(p) = \frac{\partial}{\partial t}\Big|_{(0,p)} \overbrace{f(\Theta(t,p))}^{\text{smooth}} \right.$$

$\perp$

Viceversa, given $V \in \mathfrak{X}(M)$ a smooth vector field, does it generate a global flow?

The idea is to define, given $V \in \mathfrak{X}(M)$ the collection of mapping

$$\Theta_t : M \to M \quad \text{as}$$

$$\Theta(t,p) = \Theta_t(p) := \Theta^{(p)}(t)$$

where $\Theta^{(p)}(t)$ is the integral curve of V starting at p

$\Rightarrow$ • $\Theta(0,p) = p$ ✓

• $\Theta(t, \Theta(s,p)) = \Theta_t \circ \Theta_s(p) = \Theta^{(p)}(t+s)$

$$\overset{\shortparallel}{\Theta_{t+s}}(p) = \Theta^{\Theta(s)}(t)$$

By the translation lemma

But it is not guaranteed that the domain of definition J for the integral curve of V is all of $\mathbb{R}$!

If this is the case, we say that V is complete

Example

① $V = \frac{\partial}{\partial x}$ in $\mathbb{R}^2$

$$\Rightarrow \Theta : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\Theta(t, (x, y)) = (x+t, y)$$

② consider $W = x^2 \frac{\partial}{\partial x}$ on $M = \mathbb{R}^2$

$\Rightarrow$ the integral curve for W starting at $(1, 0) = \gamma(0)$ is given by

$$\gamma(t) = \left(\frac{1}{1-t}, 0 \right)$$

the domain of definition of γ is $(-\infty, 1)$.

If the vector field V is not complete, it will generate a flow on a smaller domain

$$\mathcal{D} \subseteq \mathbb{R} \times M$$
$$\mathcal{D} = \{ (t, p) \mid t \in \mathcal{J}_{\theta^{(p)}} \}$$

- ↳ Flows allow us to talk about directional derivatives on M for vector fields
- give a very geometric explanation of $[X,Y]$

$$(\mathcal{L}_V W)_p = \frac{d}{dt}\Big|_{t=0} d(\Theta_{-t})_{\Theta_t(p)}\left(W_{\Theta_t(p)}\right)$$

where Θ is the FLOW OF V