Lecture 9

Last time

- vector fields $X : M \to TM$
  - $\to$ expression in local coords $X = \sum X^i \frac{\partial}{\partial x^i}$
  - $\to$ global examples $\partial/\partial\theta$ on S^1 $U \xrightarrow{\varphi} \hat{U} \subseteq \mathbb{R}^n_{x^1 \dots x^n}$
  - $\to$ $X : C^\infty(M) \to C^\infty(M)$ is a derivation
  - $\to$ F-related vector fields and push-forward
- Integral curves
  - $\gamma : J \subseteq \mathbb{R} \to M$ s.t. $X_{\gamma(t)} = \dot{\gamma}(t)$ $\forall t \in J$
  - Fundamental Thm of ODE $\Rightarrow$ given $X \in \mathfrak{X}(M)$

$\forall p \in M$, $\exists!$ $\gamma : J \subseteq \mathbb{R} \to M$ smooth st:

- $\gamma(0) = p$
- $X_{\gamma(t)} = \dot{\gamma}(t)$ $\forall t \in J$

Definition: A global flow on M (a 1-parameter group action) is a smooth map

$$\Theta : \mathbb{R} \times M \to M \quad \text{s.t.}$$

(1) $\Theta(0, p) = p$

(2) $\Theta(t, \Theta(s, p)) = \Theta(t+s, p)$

A collection of smooth maps

$\Theta_t : M \to M$, $\forall t \in \mathbb{R}$ s.t.

(1) $\Theta_0 = id_M$

(2) $\Theta_t \circ \Theta_s = \Theta_{t+s}$ $\Rightarrow$ $\Theta : \mathbb{R} \times M \to M$

$\Theta(t, p) = \Theta_t(p)$

Flows & vector fields

Let Θ be a global flow: $\forall\ p \in M$ we define

$\Theta^{(p)} : \mathbb{R} \longrightarrow M$ to be the smooth curve

$\Theta^{(p)}(t) := \Theta(t, p)$

Let define

$X : M \to TM$ to be

$X_p := \dot{\Theta}^{(p)}(0) \in T_pM$

Then:

- X is smooth: we use the following:

[X is smooth $\iff$ $X(f)$ is smooth $\forall f \in C^\infty(M)$]

$\Rightarrow$ we saw

$\Leftarrow$

This is a computation in local charts.

Take $f \circ \varphi^{-1} = x^i$, $x^i : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$

coord. function

$$\Rightarrow \quad X(f)(p) = \frac{\partial}{\partial t}\Big|_{(0,p)} f(\Theta(t,p))$$ (3)

└ smooth in (t,p)

∎

Viceversa

Given $X \in \mathfrak{X}(M)$ we want to obtain a flow by taking:

$$\Theta_t(p) := \Theta^{(p)}(t)$$

where $\Theta^{(p)}(t)$ is the unique integral curve of X starting at p

$$\Rightarrow \quad \bullet\ \Theta(0,p) = \Theta^{(p)}(0) = p$$

$$\bullet\ \Theta(t, \Theta(s,p)) = \Theta^{\Theta(s,p)}(t) = \Theta^{(p)}(t+s)$$

Translation lemma for int. curves

In general, however, the domain of definition of $\Theta^{(p)}$ is not $\mathbb{R}$

Example

$M = \mathbb{R}^2$, $X = x^2 \frac{\partial}{\partial x}$. We want the integral curve γ s.t.

$$\begin{cases} \gamma(0) = (1,0) \\ \dot\gamma(t) = \dot x(t)\frac{\partial}{\partial x} + \dot y(t)\frac{\partial}{\partial y} = x^2(t)\frac{\partial}{\partial x} \end{cases}$$

$\Rightarrow$ verify for $\gamma(t) = \left(\frac{1}{1-t}, 0\right)$

but $J = (-\infty, 1)$

When all the int. curves of X have domain of definition $= \mathbb{R}$, we say that X is complete.

We will see in a couple of lectures that we can use flows to define directional derivatives on manifolds not only of functions, but of any tensor.

→ covectors in linear algebra

→ covector fields or 1-forms

Covectors in Linear algebra

Let V be a finite dimensional vector space over $\mathbb{R}$. Let $v_1, \dots v_n$ be a basis for V

$$\Rightarrow \quad V \xrightarrow{\simeq} \mathbb{R}^n$$
$$v_i \longmapsto e_i$$

<u>Def</u>: A <u>covector</u> or a linear form on V is a linear map

$$\omega : V \longrightarrow \mathbb{R}$$

We denote by $V^* = \{ \omega : V \to \mathbb{R} \mid \omega \text{ covector} \}$

- V^* is a finite dimensional vector space, which we call the dual space of V
- Given any base $(e_1, \dots e_n)$ for V, there exists a <u>dual basis</u> $(\varepsilon^1, \dots \varepsilon^n)$ of V^* satisfying the property

$$\varepsilon^i(e_j) = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Kronecker δ

- Fixing $\langle e_1, \dots e_n \rangle = V$ we have that $(\varepsilon^1, \dots \varepsilon^n) = V^*$

$\to \forall v \in V, \quad v = \sum_{i=1}^{n} v^i e_i$

$\omega \in V^*$, $\omega = \sum_{i=1}^{n} \omega_i \varepsilon^i$

$$\Rightarrow \quad \omega(v) = \sum_k \omega_k v^k = (\omega_1 \ldots \omega_n) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

<u>Remark</u> (Exercise, easy)

let $(e_1, \ldots e_n)$ for V, $(\varepsilon^1, \ldots \varepsilon^n)$ for V^* dual basis. Then

→ given a vector $v \in V$ the <u>coordinates</u>/ <u>compts</u> of v in the basis $(e_1, \ldots e_n)$ are given by

$$\boxed{v^i = \varepsilon^i(v)}$$

→ For a form ω we have that

$$\omega_i = \omega(e_i)$$

→ Given $A : V \to W$ a linear map. We define the <u>dual</u> or transpose map

$A^* : W^* \to V^*$ via the rule

$$\boxed{(A^*\omega)(v) = \omega(Av)} \quad \forall \omega \in W^*, \ v \in V$$

After fixing basis for W, V and dual basis for W^*, V^*, show that A^* is given by the <u>transpose</u> of A

cotangent vectors & cotangent bundle

Definition: Let M be a smooth manifold.
We call cotangent space $T_p^* M$ of M at p the vector space

$$T_p^* M = (T_p M)^*$$

An element $\omega \in T_p^* M$ is called a cotangent vector

A very important example

Let $h: M \to \mathbb{R}$ a smooth function.

$\Rightarrow \forall p \quad (dh|_p : T_p M \to T_{h(p)}\mathbb{R} \cong \mathbb{R}$ is a cotangent vectors.

$\forall \quad v \in T_p M$

$$\boxed{dh|_p(v) = v(h)}$$

In local coordinates

Let (U, φ) a chart for M $\quad U \subseteq M$

$$\downarrow \varphi$$

Recall that we usually keep denoting with $\quad \varphi(U) \subseteq \mathbb{R}^n_{x^1, \dots, x^n}$

x^i the composition

$$\boxed{U \xrightarrow{\varphi} \hat{U} \xrightarrow{x^i} \mathbb{R}}$$

These are instead denoted by $\varphi^i : U \to \mathbb{R}$ in some exercises and in last year notes

$\Rightarrow$ $\forall\, p \in U$ we have

$$T_pM = \left\langle \frac{\partial}{\partial x^1}\Big|_p, \dots \frac{\partial}{\partial x^n}\Big|_p \right\rangle$$

$$\frac{\partial}{\partial \varphi^1}\Big|_p \quad \dots$$

$$(d_p\varphi)^{-1}(e_i|_p)$$

We want understand what the dual basis is. Notice that the coordinate functions x^i gives

$$dx^i|_p : T_pM \to T_{x^i(\varphi(p))}\mathbb{R} \simeq \mathbb{R}$$

$$\Rightarrow \boxed{dx^i|_p \in T_p^*M}$$

Moreover $\boxed{dx^i|_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p (x^i) = \delta^i_j}$

$\Rightarrow$ $\langle dx^1|_p, \text{—}\, dx^n|_p \rangle$ is the dual basis!!

$\Rightarrow$ Every $w \in T_p^* M$ can be written as

$$\sum_{i=1}^{n} w_i \, dx^i|_p \qquad \text{for } x^i : U \to \hat{U} \to \mathbb{R}$$

local coordinates

As we did for ~~tangent~~ vectors, we want to understand how the components w_i transform when we change charts

Let (U, φ), (V, ψ) two charts

$$\boxed{\begin{array}{ccc} \widehat{U\cap V} & \xrightarrow{\psi \circ \varphi^{-1}} & \widehat{U \cap V} \\ (x^1, \dots, x^n) & \longmapsto & (z^1(x^1, \dots x^n), \dots, z^n(x^1, \dots, x^n)) \end{array}}$$

$$v \in T_pM \Rightarrow v = \sum \underbrace{v^i}\frac{\partial}{\partial x^i}\Big|_p = \sum \underbrace{t^i}\frac{\partial}{\partial z^i}\Big|_p$$

$$w \in T_p^* M \Rightarrow w = \sum \underbrace{w_i}\, dx^i|_p = \sum \underbrace{\xi_i}\, dz^i|_p$$

relation ?

$$d(\psi \circ \varphi^{-1}) : T\widehat{U \cap V} \longrightarrow T\widehat{U\cap V}$$

$$\begin{pmatrix} \frac{\partial z^1}{\partial x^1} & \cdots & \frac{\partial z^1}{\partial x^n} \\ \vdots & & \\ \frac{\partial z^n}{\partial x^1} & \cdots & \frac{\partial z^n}{\partial x^n} \end{pmatrix}$$

$$\Rightarrow \quad d(\varphi\circ\psi^{-1})|_p : T_pM \longrightarrow T_pM$$

$$\begin{pmatrix} \frac{\partial z^1}{\partial x^1} & \cdots & \frac{\partial z^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial z^n}{\partial x^1} & \cdots & \frac{\partial z^n}{\partial x^n} \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial z^i}{\partial x^j} v^j \right) \frac{\partial}{\partial z^i}$$

$$\Rightarrow \quad \boxed{t^i = \sum_{j=1}^{n} \frac{\partial z^i}{\partial x^j} v^j} \qquad \frac{\partial}{\partial x^j} \longmapsto \sum_{i=1}^{n} \frac{\partial z^i}{\partial x^j} \frac{\partial}{\partial z^j}$$

Now we want to use the fact that the components ω_i and ξ_i are given respectively by

$$\boxed{\begin{aligned} \omega_j &= \omega\left(\frac{\partial}{\partial x^j}\Big|_p\right) \\ \xi_j &= \omega\left(\frac{\partial}{\partial z^j}\Big|_p\right) \end{aligned}}$$

$\Rightarrow$

from above

$$\frac{\partial}{\partial x^j}\Big|_p = \sum_{i=1}^{n} \frac{\partial z^i}{\partial x^j} \frac{\partial}{\partial z^i}$$

$$\boxed{\omega_j = \sum_{i=1}^{n} \frac{\partial z^i}{\partial x^j}\Big|_p \xi_i}$$

[compare with the exercise saying that
$A^*: V^* \to W^*$ is the transpose of $A: V \to W$]

<u>Definition</u>: the cotangent bundle $T^*M = \bigsqcup_{p \in M} T_p^*M$

<u>Proposition</u>

$T^*M \xrightarrow{\pi} M$ is a vector bundle

<u>proof</u> (<u>Exercise</u>) $\quad \sum \omega_i \, dx^i|_p \leftarrow (p, \omega = \sum \omega_i \varepsilon^i)$

- (U, φ) give trivializations $T^*M \leftarrow U \times \mathbb{R}^n$
- The computation above describe the transition function

<u>Definition</u>: A <u>section</u> $\omega: M \to T^*M$ is called a <u>covector field</u> or a <u>differential 1-form</u>.

We denote by $\Omega^1(M)$ the space of 1-form

Remark

As for $\mathfrak{X}(M)$, also $\Omega^1(M)$ is a **module** over $C^\infty(M)$, i.e. $\omega \in \Omega^1(M)$ also $f \cdot \omega$ does, $f \in C^\infty(M)$.

Definition: given $X \in \mathfrak{X}(M)$, $\omega \in \Omega^1(M)$ we define the contraction of ω with X to be the smooth function

$$\underset{i_X \omega}{\overset{\langle \omega, X \rangle}{\shortparallel}} : \begin{aligned} M &\longrightarrow \mathbb{R} \\ p &\longmapsto \omega|_p(X_p) \end{aligned}$$

smoothness can be checked in local coordinates:

$$X|_{(U,\varphi)} = \sum X^i \frac{\partial}{\partial x^i} \qquad X^i : U \to \mathbb{R} \text{ smooth}$$

$$\omega|_{(U,\varphi)} = \sum \omega_i \, dx^i \qquad \omega_i : U \to \mathbb{R} \quad \text{"}$$

$$\Rightarrow \quad i_X \omega|_U = \sum X^i \omega_i$$

Given $h : M \to \mathbb{R}$, $h \in C^\infty(M)$

$dh \in \Omega^1(M)$.

$\forall X \in \mathfrak{X}(M)$ $\boxed{dh(X)(p) = dh|_p(X_p) = X_p(h)}$

We well call a 1-form <u>exact</u> if

$\omega = dh$ for some $h \in C^\infty(M)$

<u>1-Forms and smooth maps</u>

<u>Definition</u>, Let $F : M \to N$ a smooth map

We can define the <u>pull-back</u> of 1-forms:

$$F^* : \Omega^1(N) \longrightarrow \Omega^1(M)$$

$$\omega \longmapsto F^*\omega$$

The form $F^*\omega$ acts on vector fields as follow

$\forall X \in \mathfrak{X}(M)$

$$\boxed{(F^*\omega)|_p(X_p) = \omega_{F(p)}(d_pF(X_p))}$$

Proposition

Define $F^*: C^\infty(N) \to C^\infty(M)$ to be
$$h \longmapsto h \circ F$$

$$\Rightarrow \boxed{d(F^* h) = F^* dh}$$

proof:

we test on $X \in \mathfrak{X}(M)$

$$d(F^*h)|_p(X_p) = d(h \circ F)|_p(X_p) = dh|_{F(p)}(d_pF(X_p)) = F^*(dh)(X)$$

∎

1-Form can be integrated along curves

Def: The line integral of $\omega \in \Omega^1(M)$ along $\gamma: [a,b] \subseteq \mathbb{R} \to M$ is definided by

$$\boxed{\int_\gamma \omega := \int_a^b \underbrace{\omega_{\gamma(t)}(\dot\gamma(t))}_{= \gamma^*\omega} \, dt}$$

$\gamma^*\omega$ except that $[a,b]$ is not a manifold

Theorem (1-dimensional stokes) $\gamma: [a,b] \to M$

it is a manifold with boundary

$$\int_\gamma dh = h(\gamma(b)) - h(\gamma(a))$$

proof

$$\int_\gamma dh = \int_a^b dh(\dot\gamma(t))\,dt = \int_a^b (h \circ \gamma)'(t)\,dt$$

∎