Lecture 11

Last time

- manifolds with boundary: def, examples, smooth function, tangent space
- line integrals of 1 forms and 1-dimensional Stoke's theorem
- covariant and contravariant k-tensors

$\mathrm{Ten}^k V \simeq \otimes^k V^*$ vector space of dimension n^k with base given by

$$\varepsilon^I_\otimes = \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} \qquad \varepsilon^{i_j} \in V^*$$

- let $\langle e_1, \ldots e_n \rangle$, $\langle e'_1, \ldots e'_n \rangle$ two basis for V s.t.

$$e'_j = \sum_{i=1}^{n} e^i_j e_i$$

$\Rightarrow$ If $T_I = T(e_{i_1}, \ldots e_{i_k})$

$\tilde{T}_J = T(e'_{j_1}, \ldots e'_{j_k})$

$$\Rightarrow \boxed{\tilde{T}_J = \sum_{I \in \underline{n}^k} e^{i_1}_{j_1} \cdots e^{i_k}_{j_k} \, T_I}$$

- Alternating tensors

$T \in \mathrm{Ten}^k V$ such that

$$\boxed{T(X_1, \ldots X_k) = 0} \quad \text{if } X_i = X_j \ \exists\, i \neq j$$

We denote by $Alt^k V$, $\Lambda^k V^*$ the vector space of alternating tensor

§11.1 : Base for Alternating tensors and $\wedge$ product

Definition :

We call skew-symmetrization of a tensor $T \in Ten^k$

$$(*) \quad \boxed{A(T) = \sum_{\sigma \in S_k} sgn(\sigma)\, \sigma T} \in Alt^k V$$

Example

On $\mathbb{R}^2$, let $\varepsilon^1, \varepsilon^2$ the dual of the standard basis. Then the general element of $Ten^2 \mathbb{R}^2$ is written as

$$T = T_{11}\, \varepsilon^1 \otimes \varepsilon^1 + T_{12}\, \varepsilon^1 \otimes \varepsilon^2 + T_{21}\, \varepsilon^2 \otimes \varepsilon^1 + T_{22}\, \varepsilon^2 \otimes \varepsilon^2$$

$$A(T) = T - T_{11}\, \varepsilon^1 \otimes \varepsilon^1 - T_{12}\, \varepsilon^2 \otimes \varepsilon^1 - T_{21}\, \varepsilon^1 \otimes \varepsilon^2 - T_{22}\, \varepsilon^2 \otimes \varepsilon^2$$

$$= (T_{12} - T_{21})\, \varepsilon^1 \otimes \varepsilon^2 + (T_{21} - T_{12})\, \varepsilon^2 \otimes \varepsilon^1$$

A base for alternating tensors

Riemannian metric

Definition : We define elementary alternating tensor and denote by ε^I

* you can also define a symmetric tensor, i.e. : $T = \sigma^* T \ \forall \sigma \in S_k$; $Sym T = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma T$

the alternating k-tensors on V defined by

$$\boxed{\varepsilon^I = A(\varepsilon^I_{\otimes}) = \sum_{\sigma \in S_k} sg(\sigma)\, \varepsilon^{i_{\sigma(1)}} \otimes \cdots \otimes \varepsilon^{i_{\sigma(k)}}}$$

In particular

given $v_1, \dots v_k \in V$ vectors

$$\Rightarrow \boxed{\varepsilon^I(v_1, \dots v_k) = \det \begin{pmatrix} v_1^{i_1} & \dots & v_k^{i_1} \\ \vdots & & \vdots \\ v_1^{i_k} & & v_k^{i_k} \end{pmatrix}}$$

where $v_j^{i_m} = \varepsilon^i(v_j)$

Lemma

(a) $\varepsilon^I = 0$ if I has a repeated index

(b) if $J = \sigma^* I \Rightarrow \varepsilon^J = sgn(\sigma)\, \varepsilon^I$

(c) $\varepsilon^I(E_J) = \begin{cases} 0 & \text{if } J \text{ has a repeated index or } J \neq \sigma^* I \\ sgn(\sigma) & J = \sigma^* I \end{cases}$

proof (Exercise)

Def: we define $I = (i_1 \dots i_k)$ an increasing multiindex if $i_1 < i_2 \dots < i_k$

Let $\underline{n}^{k\nearrow}$ denote the set of increasing multiindeces. $\Rightarrow$ $|\underline{n}^{k\nearrow}| = \binom{n}{k}$

__Proposition__

A base for $Alt^k V = \Lambda^k V^*$ is given by the elementary alternating tensor ε^I, $I \in \underline{n}^k\nearrow$. In particular $\boxed{\dim \Lambda^k V^* \text{ is } \binom{n}{k}}$.

$\Leftrightarrow$ $T \in \Lambda^k V^*$

i.e.

$$T = \sum_{I \in \underline{n}^k\nearrow} T_I \, \varepsilon^I$$

In particular $Alt^n V = \Lambda^n V^*$ is a 1-dimensional vector space!

↳ up to scaling it is generated by the determinant.

__Definition__: The __wedge product__ or __exterior product__ of $S \in Alt^k V$, $T \in Alt^\ell V$ is defined as

$$\boxed{S \wedge T = \frac{1}{k!\,\ell!} A(S \otimes T)}$$

$$\boxed{T_1 \wedge \dots \wedge T_m = \frac{1}{k_1! \dots k_m!} A(T_1 \otimes \dots \otimes T_m)}$$

Examples

- $c \in Alt^0 V = \mathbb{R}$, $T \in Ten^k V$

$\Rightarrow \quad c \wedge T = cT$

- $\alpha, \beta \in Alt^1 V = V^*$

$\alpha \wedge \beta \ (X, y) = \alpha(X)\beta(y) - \alpha(y)\beta(X)$

- $\omega \in Alt^2(V), \sigma \in Alt^1(V) = V^*$

$$\sigma \wedge \omega \ (x_1, x_2, X_3) = \frac{1}{2} (\omega(X_1, X_2)\sigma(X_3) - \omega(X_1, X_3)\sigma(X_2) + \omega(X_2, X_3)\sigma(X_1) - \omega(X_2, X_1)\sigma(X_3) + \omega(X_3, X_1)\sigma(X_2) - \omega(X_3, X_1)\sigma(X_1) =$$

Alternating

$$= \omega(X_1, X_2)\sigma(X_3) - \omega(X_1, X_3)\sigma(X_1) + \omega(X_2, X_3)\sigma(X_1)$$

- $\boxed{\varphi^1 \wedge \ldots \wedge \varphi^k (X_1, \ldots X_k) = \det \left(\varphi^i(X_j)_{i,j=1}^{k} \right)}$

Proposition

The wedge product satisfy the following

(a) $\wedge : \Lambda^k V^* \times \Lambda^l V^* \to \Lambda^{k+l} V^*$ is associative $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$

(b) $\wedge$ is an anticommutative product, i.e.

$$\omega_1 \wedge \omega_2 = (-1)^{\deg \omega_1 \cdot \deg \omega_2} \omega_2 \wedge \omega_1$$

(c) if $\varepsilon^{i_1}, \dots \varepsilon^{i_k}$ are elements in V^*

$\Rightarrow \varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_k}$ is ε^I the elementary alternating tensor

(e) $\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}$ $\quad IJ = (I, J)$

proof [Exercise]

§ 11.2 Tensor field and differential form on smooth manifolds

Let M be a differentiable manifold

Definition: we denote by

$$\Lambda^k T^* M = \bigsqcup_{p \in M} \Lambda^k T_p^* M$$

Proposition

$\Lambda^k T^* M \xrightarrow{\pi} M$ is a rank $\binom{n}{k}$ vector bundle on M

proof: (Exercise)

$\Rightarrow$ let (U, φ) a smooth chart for M

$$U \xrightarrow{\varphi} \varphi(U) \subseteq \mathbb{R}^n_{x^1, \dots x^n}$$

$$\varphi^i = x^i \circ \varphi$$

We have already seen that

$$TM|_U \xrightarrow[\cong]{\Phi_U} U \times \mathbb{R}^n \qquad \phi_U \text{ diff}$$

$$\sum_i v^i \frac{\partial}{\partial \varphi^i}\Big|_p \longleftarrow (p, \sum v^i e_i)$$

$$T^*M|_U \longleftrightarrow U \times (\mathbb{R}^n)^*$$

$$\sum w_i \, d\varphi^i|_p \longleftarrow (p, \sum w_i \varepsilon^i)$$

$$\boxed{\begin{array}{l} \Lambda^k T^* M|_U \longmapsto (p, \Lambda^k (\mathbb{R}^n)^*) \\ \left(\sum_{I \in \eta^k} w_I \, d\varphi^{i_1} \wedge \dots \wedge d\varphi^{i_k} \right) \longmapsto (p, \sum_{I \in \eta^k} w_I \, \varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_k}) \end{array}}$$

$\Rightarrow$ Work out how the w_I transform under change of coordinates!

Definition

A differential k-form or a k-form on M is a smooth section of $\Lambda^k T^*M$, i.e.

$$\boxed{\omega : M \longrightarrow \Lambda^k T^*M}$$

Given a coordinate chart (U, φ) of M, ω can be written as follow

$$\boxed{\omega|_U = \sum_{I \in \underline{n}^k_{\nearrow}} \omega_I \, d\varphi^{i_1} \wedge \dots \wedge d\varphi^{i_k}}$$

where $\omega_I : U \longrightarrow \mathbb{R}$ are smooth functions. These are called the component functions of ω w.r. to (U, φ)

Remark For any multiindex $J \in \underline{n}^k$ the component function

$$\boxed{\omega_J|_U = \omega\left(\frac{\partial}{\partial \varphi^{j_1}}, \dots, \frac{\partial}{\partial \varphi^{j_k}}\right)}$$

- We denote by $\Omega^k(M)$ the set of k-forms. This is a module on $C^\infty(M)$
i.e. given $f, g \in C^\infty(M)$, $\omega_1, \omega_2 \in \Omega^k(M)$

$$f\omega_1 + g\omega_2 \in \Omega^k(M)$$

- Given $\omega \in \Omega^k(M)$ defines a multilinear map

$$\omega : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$$

$$X_1, \dots X_k \longmapsto \omega(X_1, \dots X_k)$$

In local coordinates

$$\omega = \sum_{I \in \underline{n}^k_{\nearrow}} \omega_I \, d\varphi^{i_1} \wedge \dots \wedge d\varphi^{i_k} \qquad X_j = \sum_i X_j^i \frac{\partial}{\partial \varphi^i}$$

$$= \boxed{\omega(X_1, \dots X_k) \equiv \sum_{I \in \underline{n}^k} \omega_I \cdot \det \left(\varphi^{i_j}(X_j)\Big|_{j=1}^{k} \right)}$$

- Given $\alpha \in \Omega^k(M)$, $\beta \in \Omega^\ell(M)$

$$\boxed{\alpha \wedge \beta \in \Omega^{k+\ell}(M)}$$

In local coordinates

$$\alpha = \sum_I \alpha_I \, d\varphi^I \qquad \beta = \sum_J \alpha_J \, d\varphi^J$$

$$\alpha \wedge \beta = \sum_{\substack{I, J \\ I \cap J = \emptyset}} \alpha_I \beta_J \, d\varphi^{IJ}$$

- Given $X \in \mathfrak{X}(M)$, $\omega \in \Omega^k(M)$

$$\boxed{\begin{array}{l} i_X(\omega) \in \Omega^{k-1}(M) \\ X \lrcorner \omega \end{array}}$$

show they take the same value on each $X_1, \dots, X_k$

In local coordinates

$$\omega = \sum_I \omega_I \, d\varphi^{i_1} \wedge \dots \wedge d\varphi^{i_k}$$

$$\Rightarrow \boxed{X \lrcorner \omega = \sum_I \omega_I \sum_{j=1}^{k} (-1)^{j-1} X_{i_j} \, d\varphi^{i_1} \dots \wedge \widehat{d\varphi^{i_j}} \dots d\varphi^{i_k}}$$

- Pull-back

Let $F: M \to N$ smooth

$$\Rightarrow F^*: \Omega^k(N) \longrightarrow \Omega^k(M)$$

defined by

$$\boxed{F^*\omega : \mathfrak{X}(M)^k \longrightarrow C^\infty(M)}$$

$$F^*\omega(X_1, \dots X_k)(p) = \omega_{F(p)}\big(d_pF(X_{1}(p)) \dots d_pF(X_{k}(p))\big)$$

Claim $\boxed{F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta}$ prove it

Given A local presentation of $M \xrightarrow{F} N$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & N \supseteq V \arrow[d, "\psi"] \\ \arrow[r, "\hat{F}"] & \end{tikzcd}$$

$x^1, \dots x^n$ $\qquad$ $y^1, \dots y^k$

$$y^i = \hat{F}_i(x^1, \dots x^n)$$

$\Rightarrow$ given $\omega = \sum \omega_I \, dy^{i_1} \wedge \dots dy^{i_k}$

$$f^* \omega = \sum \omega \circ F \; d(\hat{F}_{i_1}) \wedge \dots \wedge d(\hat{F}_{i_k})$$

__Example__ : $F : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad (r, \theta) \to (r\cos\theta, r\sin\theta)$

let us compute

$$F^*(dx \wedge dy) = F^* dx \wedge F^* dy$$

$$= d(r\cos\theta) \wedge d(r\sin\theta)$$

$$= (\cos\theta \, dr - r\sin\theta \, d\theta)$$

$$\wedge (\sin\theta \, dr + r\cos\theta \, d\theta) =$$

$$= r\cos\theta^2 \, dr \wedge d\theta - r\sin^2\theta \, d\theta \wedge dr$$

$$= \underline{r \, dr \wedge d\theta}$$

Lemma: let $F: M \to N$ a smooth map between n-dimensional manifolds

let (U, φ) a chart for M, (V, ψ) a chart for N

$\Rightarrow$ on $U \cap F^{-1}(V)$ we have that

$$\boxed{F^*(u \, d\psi^1 \wedge \dots \wedge d\psi^n) = u \circ F \cdot \det\left(\frac{\partial F^j}{d\varphi^i}\right) d\varphi^1 \wedge \dots \wedge d\varphi^n}$$

change of coordinates

Proof

- since $\Lambda^n T^*M$ has rank 1, i.e.

$\Lambda^n T^*M|_U = U \times \mathbb{R}$ it is true that any n-form on U can be written as $g \cdot d\varphi^1 \wedge \dots \wedge d\varphi^n$ $\quad g \in C^\infty(U)$

- to check that ω_1, ω_2 coincide, it is enough to see that

$$\omega_1\left(\frac{\partial}{\partial \varphi^1}, \frac{\partial}{\partial \varphi^n}\right) = \omega_2(\qquad)$$

$$F^*(v\, d\varphi^1 \wedge \dots \wedge d\varphi^n)\left(\frac{\partial}{\partial \varphi^1}, \dots, \frac{\partial}{\varphi^n}\right) =$$

$$= v\circ F \; d\varphi^1 \wedge \dots \wedge d\varphi^n \left(D_p F\left(\frac{\partial}{\partial \varphi^1}\right), \dots D_p F\left(\frac{\partial}{\partial \varphi^n}\right)\right)$$

$$\underset{\displaystyle \sum_{j=1}^{n} \frac{\partial F_j}{\partial \varphi^1}\frac{\partial}{\partial \varphi^j}}{\parallel}$$

$$= v\circ F \quad \det\left(\left(\frac{\partial F_j}{\partial \varphi^i}\right)_{i,j=1,\dots n}\right)$$

$$= v\circ F \cdot \det\left(\frac{\partial F_j}{\partial \varphi^i}\right) \; d\varphi^1 \wedge \dots d\varphi^j \left(\frac{\partial}{\partial \varphi^1}, \dots \frac{\partial}{\partial \varphi^j}\right)$$

- Exterior derivative

Let denote by $\Omega^0(M) = C^\infty(M)$. We have seen that $\forall$ $f \in \Omega^0(M)$ $df \in \Omega^1(M)$

We can define an operator

$$\boxed{d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)}$$

In local coordinates

$$\omega = \sum_{I \in n^k \nearrow} \omega_I \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$\Rightarrow \boxed{d\omega = \sum_I \sum_{j=1}^{n} \frac{\partial \omega_I}{\partial x^j} dx^I \wedge dx^j}$$

ie

$$\boxed{d\omega = \sum_I d\omega_I \wedge dx^I}$$

Proposition [14.23, 14.24 Lee's Book]

(a) d is $\mathbb{R}$-linear

(b) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$

(c) $\boxed{d \circ d \equiv 0}$

(d) $F^*(d\omega) = d(F^*\omega)$

proof : use the coordinate expression