Exercise Series 12 - Exterior differentials, Orientations, Integration on Manifolds **2022–12–05**

Exercise 12.1. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the smooth map $F(\theta, \varphi) = ((cos\varphi{+}2)cos\theta, (cos\varphi{+}2)sin\theta, sin\varphi)$. Let consider $\omega = ydz \wedge dx$. Compute $d\omega$ and $F^*\omega$ and verify by direct computation that $d(F^*\omega) = F^*d\omega$.

Exercise 12.2. Prove that given $\omega \in \Omega^k(\mathbb{R}^n)$ a k-form the Exterior derivative $d\omega = d(\sum_I \omega_I dx^I) := \sum_I d\omega_I \wedge dx^I$ has the following properties:

(a) d is $\mathbb{R}$-linear
(b) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge d\eta$
(c) $d \circ d = 0$
(d) Given $F\colon U \to V$ a smooth map $d(F^*\omega) = F^*d\omega$

Exercise 12.3. Suppose that M is a smooth manifold which is the union of two orientable open sub-manifolds with connected intersections. Show that then M is orientable. Use this to prove that S^n is orientable.

Exercise 12.4. * Let M a smooth manifold. Show that TM and T^*M are always orientable manifolds.

Recall that a smooth covering map $\pi\colon E \to M$ is a smooth surjective morphism such that for each $p \in M$ there exists a open neighbourhood U in M with $\pi^{-1}(U) = \sqcup_{i=1}^k V_i$ with $\pi|_{V_i}\colon V_i \to U$ is a diffeomorphism.

The following two results explain how orientation behave with respect to smooth covering maps and give us criteria to show when a manifold is not orientable.

Theorem Let E be connected, oriented, smooth manifolds and $\pi\colon E \to M$ be a smooth normal covering map.[1] Then M is orientable if and only if the action of $\mathrm{Aut}_\pi(E)$ is orientation preserving

Exercise 12.5. * Using the above theorem, prove that $\mathbb{RP}^n$ is orientable if and only if n is odd.

Hint Consider $q :: S^n \to \mathbb{RP}^n$ the quotient map. Prove that this is a normal smooth covering map. Then prove that the only non trivial automorphism of q is the antipodal map $x \mapsto -x$ which is orientation preserving if and only if n is odd.

[1]For us normal will just means that the group $\mathrm{Aut}_\pi(E)$ of automorphism of E commuting with π acts transitively on the fiber.