Lecture 12

Last time

→ the bundle $\Lambda^k T^* M$ and its sections $\omega : M \to \Lambda^k T^* M$

differential k-forms ; $\Omega^k(M)$ space of k-forms

→ (U, φ) coordinate chart for M

$$\omega|_U = \sum_{I \in \mathcal{I}^k_{\nearrow}} \omega_I \, d\varphi^{i_1} \wedge \dots \wedge d\varphi^{i_k}$$

where $\omega_I = \omega\left(\frac{\partial}{\partial \varphi^{i_1}}, \dots \frac{\partial}{\varphi^{i_k}}\right)$

→ $\Omega^k(M) = \{ \omega : \mathfrak{X}(M)^k \to C^\infty(M) \mid \omega \text{ is alternating} \}$

→ $X \in \mathfrak{X}(M)$, $\omega \in \Omega^k(M)$

$X \lrcorner \omega \in \Omega^{k-1}(M)$

→ given $F : M \to N$ smooth we have

$F^* : \Omega^k(N) \to \Omega^k(M)$

Lemma

Let $F : M \to N$ be smooth map between n-dimensional manifolds. Let $(U, \varphi) \subseteq M$, $(F(U) \subseteq V, \psi) \subseteq N$ smooth charts

$$\Rightarrow \boxed{F^*(u \, d\psi^1 \wedge \dots \wedge d\psi^n) = u \circ F \det\left(\frac{\partial F^j}{\partial \varphi^i}\right) d\varphi^1 \dots d\varphi^n}$$

→ Exterior derivative

There exists an operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ which extends the usual differential of smooth functions $d : C^\infty(M) = \Omega^0(M) \to \Omega^1(M)$

Let $M = \mathbb{R}^n$

↳ $\omega \in \Omega^k(\mathbb{R}^n)$ $\qquad \omega = \sum_I \omega_I \underbrace{dx^{i_1} \wedge \dots \wedge dx^{i_k}}_{dx^I}$

We then define

$$\boxed{d\omega := \sum_I d\omega_I \wedge dx^I}$$

Proposition (Exercise)

(a) d is $\mathbb{R}$-linear

(b) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$

(c) $d \circ d = 0$

(d) $F^* d\omega = d(F^*\omega)$

Hint : After having observed a , it is enough to prove the statements for forms

$$\omega = f\, dx^{i_1} \wedge \_ \, dx^{i_k} \qquad f \in C^\infty(\mathbb{R}^n)$$

Theorem (14.24 Lee's book)

Let M be a smooth manifold

$\Rightarrow$ $\exists !$ $d : \Omega^k(M) \to \Omega^{k+1}(M)$

having the property (a), —(d) above

proof [existence]

Given a chart $(U, \varphi) \subseteq M$ we want to define

$$d\omega|_U := \varphi^*\left(d(\varphi^{-1})^*\omega\right)$$

we need to show that the definition is well posed, $\Leftrightarrow$ does not depend from the choice of coordinates (so we can give!)

Let (U', ψ) another chart

$$\begin{tikzcd} (U\cap V) \arrow[d, bend right, "\varphi"'] & (U\cap Y) \arrow[d, "\psi"] \\ \varphi(U\cap U') \arrow[u, bend left, "\varphi^{-1}"] \arrow[r, "\psi\circ\varphi^{-1}"] & \psi(U\cap U') \arrow[u, bend right, "\psi^{-1}"'] \end{tikzcd}$$

$$\varphi^*\left(d(\varphi^{-1})^*\omega)\right) = \varphi^* d\left((\psi^{-1}\circ\psi\circ\varphi^{-1}\right)^*\omega =$$

$$= \varphi^* d\left(\varphi^{-1*}\circ\psi^*\circ(\psi^{-1})^*\omega\right) =$$

$$= \varphi^*\circ(\varphi^{-1})^*\circ\psi^* \; d(\psi^{-1})^*\omega\Big) \qquad \square$$

§ 12.1 Orientation on Manifolds

Definition: Let V be a vector space of dim n. We say that two bases $\langle e_1, \_ e_n\rangle$, $\langle f_1, \_ f_n\rangle$ are consistently oriented if the matrix of the change of base $B = (B_i^j)_{i,j}$ s.t. $e_i = \sum B_i^j f_j$ satisfy

$$\boxed{\det B > 0}$$

An orientation for V is an equivolence class of of ordered basis *

$\Leftrightarrow$ $\mathcal{O} : \{\text{Basis of } V\} \longrightarrow \{\pm 1\}$

$$\langle b_1, \_ b_n \rangle \rightarrow \text{sgn}$$

Example on $\mathbb{R}^n$ we have a standard orientation $\mathcal{O}_{std}$ induced by the equivolence class of the standard basis $\langle e_1, \_ e_n \rangle$

$$\mathcal{O}_{std} : \{\text{Basis of } \mathbb{R}^n\} \longrightarrow \{\pm 1\}$$

$$\langle b_1, \_ b_n \rangle \longmapsto \text{sgn}(\det B)$$

↳ An orientation $\mathcal{O}$ on V is determined by it's value at a base $\mathcal{B}$

Indeed, $\forall$ other base $\mathcal{B}'$, $\exists$ $C \in GL(\dim V, \mathbb{R})$

$$\mathcal{B}' = C \cdot \mathcal{B} \quad \Rightarrow \quad \boxed{\mathcal{O}(\mathcal{B}') = \text{sgn}(\det(C))\, \mathcal{O}(\mathcal{B})}$$

↳ $\exists$ exactly two possibilities : $\mathcal{O}(\mathcal{B}) = 1 \leftrightarrow \mathcal{O}$

$\mathcal{O}(\mathcal{B}) = -1 \leftrightarrow -\mathcal{O}$

Important exemple

Fixing $\omega \in \Lambda^n V^*$ defines an orientation $\mathcal{O}_\omega$ on V by setting

$$\mathcal{O}_\omega(B) = \operatorname{sgn}(\omega(B_1, \_ B_n))$$

Given $(V, \mathcal{O}^V)$, $(W, \mathcal{O}^W)$ two oriented vector spaces of dim n and $T : V \to W$ an isomorphism. Let $\langle b_1, \_ b_n \rangle$ a base of V and $\langle T(b_1), \_ T(b_n) \rangle$ a base of V

$$\Rightarrow \quad \mathcal{O}^W(T(B)) = \operatorname{sgn}(\det T^1_B) \cdot \mathcal{O}^V(B)$$

We say that T is orientation preserving if $\operatorname{sgn} T = 1$ and orientation reversing if $\operatorname{sgn} T = -1$.

Definition, let M be an n-dimensional manifold. An orientation $\mathcal{O}$ on M is a function that to each p assigns $\mathcal{O}_p$ an orientation on T_pM $\quad \mathcal{O} : M \to \{\pm 1\}$

$$\mathcal{O} \to \mathcal{O}_p : T_pM \to \{\pm 1\}$$

We say that $\mathcal{O}$ is continuos if $\forall\ p \in M\ \exists$ a chart (U, φ) around p such that

The Function $\mathcal{O}: U \longrightarrow \{\pm 1\}$

$$\boxed{p \longmapsto \mathcal{O}_p\left(\frac{\partial}{\partial\varphi^1}\Big|_p, \ldots, \frac{\partial}{\partial\varphi^n}\Big|_p\right)}$$

is constant on $p \in U_p \subseteq U$.

If M admits a continuos orientation $\Rightarrow$ we say that M is orientable

We say that $(M, \mathcal{O})$ is an oriented manifold.

Important Remark

explicit on what this is

- For $\mathcal{O}$ a continuos orientation, we have that

$$\boxed{\begin{aligned}\mathcal{O}_p &:= \mathcal{O}_p\left(\frac{\partial}{\partial\varphi_1}\Big|_p, \ldots, \frac{\partial}{\partial\varphi_n}\Big|_p\right) \\ &= \operatorname{sgn}\left(D_p\varphi : (T_pM, \mathcal{O}) \longrightarrow (\mathbb{R}^n, \mathcal{O}_{std})\right)\end{aligned}}$$

- If (U', ψ) is another chart around p

$$\Rightarrow \boxed{\mathcal{O}_p(\psi) = \operatorname{sgn} \det D_{\varphi(p)}(\psi \circ \varphi^{-1})\, \mathcal{O}_p(\varphi)}$$

is constant on a connected nbd of p

$\Rightarrow$ $\mathcal{O}_p(\psi)$ is constant on a nbd

$\Leftrightarrow$ $\mathcal{O}_p(\varphi)$ is constant on a neighborhood

⇒ The definition of continuity on $\mathcal{O}$ is well posed, <u>does not</u> depend on the choice of chart

<u>Definition</u>: let $\mathcal{A}=\{(U_\alpha,\varphi_\alpha)\}$ a smooth atlas for M. We define a <u>orientation-signed</u> atlas to be $\mathcal{A}$ together with

$$s: \mathcal{A}\to\{\pm1\}$$

$$(U_\alpha,\varphi_\alpha)\mapsto s(\varphi_\alpha)=\mathcal{O}_p\left(\frac{\partial}{\partial\varphi^\alpha}\Big|_p\right)$$

$$=\operatorname{sgn}(D_p\varphi)$$

⇒ s satisfy

$$s(\psi)=\operatorname{sgn}\det(D_{\varphi(p)}\,\psi\circ\varphi^{-1})\,s(\varphi)$$

A smooth atlas is said to be consistently oriented if $\varphi_\beta\circ\varphi_\alpha^{-1}$ has positive determinant all of $\varphi_\alpha(U_\alpha\cap U_\beta)$

<u>Proposition</u> (15.6 Lee, 7.1.8 Notes)

Let M a smooth manifold.

1) Let $(\mathcal{A},s)$ a consistently oriented atlas

$\Rightarrow \exists!\ \mathcal{O}$ orientation on M s.t. $\mathcal{O}_p(\varphi_\alpha)>0\ \forall\varphi_\alpha$

2) Viceversa, If M is oriented

⇒ The collection of (~~positively~~) oriented smooth charts is a consistently oriented atlas.

<u>Proof</u>

(1) Let (A, s) consistently oriented, then for (U, φ) a chart, define $\mathcal{O}_p(\varphi)$ to be the unique orientation on T_pM s.t. $D_p\varphi$ has positive sgn

↳ This glues because on the overlaps the sign is the same.

(2) If M is oriented, given any chart, either φ or $-\varphi$ is positively oriented. By choosing all the charts in such a way that they are positively oriented, their transition function will have positive determinant. ∎

Proposition (15.5 Lee; 7.1.1 Notes)

Let M be a smooth n-dimensional

(1) manifold. Let $\omega \in \Omega^n(M)$ a nowhere vanishing form. Then ω defines an orientation

$$\mathcal{O}_p M \to \{\pm 1\} \text{ by}$$

$$p \longmapsto \mathcal{O}_p(v_1, \_ v_n) = \operatorname{sgn}(\omega(v_1, \_ v_n))$$

(2) Every orientation on a smooth manifold is the sgn of a nowhere vanishing n-form.

proof

(1) we only need to check continuity of

$$\mathcal{O}_p(v_1, \_ v_n) := \operatorname{sgn} \omega_p(v_1 \_ v_n)$$

$\Rightarrow \exists\ \forall p\ (U, \varphi)$ s.t. $\operatorname{sgn} \omega_p\left(\frac{\partial}{\partial \varphi^1}\Big|_p, \_ \frac{\partial}{\partial \varphi^n}\Big|_p\right)$ is constant.

on U, $\omega = f\, d\varphi^1 \wedge \_ d\varphi^n$ f continuos

$\Rightarrow$ up to shrinking U, we can assume it is connected $\Rightarrow$ since $f \neq 0\ \forall p \in U$ by hp $\Rightarrow$ sgn(f) is constant on U

$$\Rightarrow \operatorname{sgn} \omega\left(\frac{\partial}{\varphi^1}, \frac{\partial}{\varphi^n}\right) = \operatorname{sgn} f \text{ is const in } U$$

(2) Viceversa, we want to construct ω

Up to passing to a refinement of $A = (\{U, \varphi\})$ we can assume all U are connected.

$\Rightarrow$ If $\mathcal{O}$ is a continuous orientation

$$\Rightarrow \quad \mathcal{O} : U \longrightarrow \{\pm 1\}$$

$$p \mapsto \mathcal{O}_p\left(\frac{\partial}{\partial \varphi^1}\Big|_p, \ldots \frac{\partial}{\partial \varphi^n}\Big|_p\right) = \mathcal{O}_p(\varphi)$$

is <u>constant</u>.

$\Rightarrow \forall \varphi$ we can define a non vanishing form $\boxed{\omega_\varphi = \mathcal{O}(\varphi)\, d\varphi^1 \wedge \ldots \wedge d\varphi^n.}$

$\Rightarrow$ Choose $(\eta_\varphi)_{\varphi \in A}$ a partition of unity subordinated to $\{U_\varphi\}_{\varphi \in A}$

$\Rightarrow$ We can define a global n-form

$$\boxed{\omega = \sum_{\varphi \in A} \eta_\varphi \omega_\varphi}$$

We are left to check that ω is not vanishing anywhere and that

$$\mathcal{O}_p = \operatorname{sgn} \omega_p$$

let $\langle b_1, - b_n \rangle = T_pM$

$$\omega|_p(b_1, - b_n) = \sum_{\varphi \in A} \underbrace{\eta_\varphi(p)}_{\geq 0} \underbrace{\omega_\varphi|_p(b)}_{\text{have sgn } \mathcal{O}_p(b)\ \forall \varphi}$$

and not all 0 since they sum to 1 at p

$\Downarrow$ $\blacksquare$

M is orientable $\Longleftrightarrow$ it admits a volume form, i.e. $\omega \in \Omega^n(M)$ nowhere vanishing

Important example

$S \hookrightarrow (M, \mathcal{O})$ an embedded hypersurface in an oriented n+1 dimensional manifold

$\Rightarrow$ A transverse vector field Y on S [i.e. $Y : S \to TM \mid Y_p \in T_pM \setminus T_pS \ \forall p \in S$] induces an orientation

$$\mathcal{O}^Y_p(X_1, - X_n) = \mathcal{O}_p(Y, X_1, - X_n)$$

Example on S^2, take $N = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$

$\Rightarrow$ N is transverse for S^2

What is the associated volume form?

§ 12.2 Integration

Let M be a smooth manifold, $\omega \in \Omega^n(M)$

$$\Rightarrow \operatorname{supp}\omega = \overline{\{p \in M \mid \omega_p \neq 0\}}$$

Definition: Let $(M, \mathcal{O})$ an oriented mfd, $\omega \in \Omega^n(M)$ s.t. $\operatorname{supp}\omega$ is compact and $\operatorname{supp}\omega \subseteq (U, \varphi^{-1})$ coordinate chart

$\Rightarrow$ The integral of ω is defined to be

$$\boxed{\int_M \omega := \operatorname{sgn}_{\mathcal{O}}\varphi \int_U \varphi^*\omega = \operatorname{sgn}\varphi \int_{\varphi(U)} h \, dx_1 \dots dx^n}$$

Proposition

The integrals is independent from the choise of chart.

proof

- Let coll $\varphi : U' \subseteq \mathbb{R}^n \to M$, $\psi : V \overset{\mathbb{R}^n}{\subseteq} \to M$.

Up to shrinking U', V if necessary, we can assume that

$$\varphi(U) = \psi(V).$$

◦ write $\sigma : \varphi^{-1} \circ \psi : V \to U$, this is a diffeomorphism because it's a change of coord. And

$$\boxed{\operatorname{sgn}\psi = \operatorname{sgn}\varphi \cdot \operatorname{sgn}(\det D\sigma)}$$

◦ Write $\varphi^*\omega = h\, dx^1 \wedge \text{—} dx^n$

$\psi^*\omega = g\, dy^1 \wedge \text{—} \wedge dy^n \Rightarrow \boxed{g = h(\sigma(y)) \det D(\sigma)}$

$$\Rightarrow \operatorname{sgn}\varphi \int_U \varphi^*\omega = \operatorname{sgn}\varphi \int_U h(x)\, dx^1 \text{—} dx^n =$$

change of variable $$= \operatorname{sgn}\varphi \int_V h(\sigma(y)) \| \det D_x \sigma \| \, dy^1 \text{—} dy^n$$

$\operatorname{sgn}^2 = 1$ $$= \operatorname{sgn}\psi \cdot \operatorname{sgn}(\det D_x \sigma) \int_V h(\sigma(y)) \det D_x \sigma \, dy$$

$$= \operatorname{sgn}\psi \int_V g\, dy^1 \text{—} dy^n$$

$$= \operatorname{sgn}\psi \cdot \int_V \psi^*(\omega) \qquad \blacksquare$$

On $\mathbb{R}^n$, φ the standard coord, the orientation has sgn $= +1$

Definition : $(M, \mathcal{O})$ oriented.

Let $\omega \in \Omega^n(M)$ s.t. $\operatorname{supp}\omega$ is compact

Let $\{(U_i, \varphi_i^{-1})\}$ charts of a consistently oriented atlas covering $\operatorname{supp}\omega$, and

Let $(\chi_i)_i$ be a partition of unity subordinate to U_i

$$\Rightarrow \quad \int_{(M,\mathcal{O})} \omega = \sum_i \int_{(M,\mathcal{O})} \chi_i \omega$$

where $$\int_M \chi_i \omega = \operatorname{sgn}(\varphi_i) \int_{\varphi_i^{-1}(U_i)} \varphi_i^*(\chi_i \omega)$$

Proposition

The above definition is well posed. it does not depend on the choice of charts nor on the partition of unity

PROOF

$$\sum_J \int_M \zeta_J \omega \overset{\chi_i\text{ - part of unity}}{=} \sum_J \int_M \sum_i \chi_i \zeta_J \omega \overset{\text{finite sum com with integral}}{=} \sum_{iJ} \int_M \chi_i \zeta_J \omega)$$

do not depend from the chart!