Lecture 13

Last time

- Exterior differential $d: \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying

(0) d on $\Omega^0(M) = C^\infty(M)$ is the usual diff.

(1) d is $\mathbb{R}$-linear

(2) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$

(3) $d \circ d = 0$

(4) $F^*(d\omega) = d(F^*\omega)$

In local coordinates $\omega = \sum \omega^I dx^I$

$$\Rightarrow d\omega = \sum_I d\omega^I \wedge dx^I$$

- Let V a vector space an orientation

$$\mathcal{O} : \{\text{Basis of } V\} \to \{\pm 1\}$$

s.t. for $B' = CB$ $\quad \mathcal{O}(B') = \mathrm{sgn}(\det(C))\, \mathcal{O}(B)$

$\mathcal{O} \Leftarrow \omega \in \Lambda^n V^*$

$$\mathcal{O}_\omega(B) = \mathrm{sgn}(\omega(b_1, \dots, b_n))$$

- Let M be a mfd. A orientation $\mathcal{O}$ on M is the data of an orientation $\mathcal{O}_p$ on T_pM $\forall p$

$\mathcal{O}$ is called continuos if

$\forall p \in M \quad \exists \; (U, \varphi)$ s.t.

$$\underbrace{\mathcal{O}_p\left(\frac{\partial}{\partial\varphi^1}\Big|_p, \dots, \frac{\partial}{\partial\varphi^n}\Big|_p\right) \text{ is constant}}$$

in a nbd of p

independent from choice of chart

Rmk

① If M is covered by a single chart
$\Rightarrow M$ is orientable

in this case $d\varphi : TM \longrightarrow \varphi(M) \times \mathbb{R}^n$

$\Rightarrow \forall p \quad d_p\varphi : T_pM \longrightarrow \mathbb{R}^n$

$\Rightarrow \quad \mathcal{O}_p\left(\frac{\partial}{\partial\varphi^1}, \dots, \frac{\partial}{\partial\varphi^n}\right) := \pm \; \operatorname{sgn}(\det d_p\varphi)$

② If we have two connected charts $(U, \varphi), (V, \psi)$ covering M, then M is orientable

$\Leftrightarrow \forall p \in U \cap V \longrightarrow \operatorname{sgn}\left(D_{\varphi(p)} \psi \circ \varphi^{-1}\right)$ is constant.

this is always the case for $U \cap V$ connected, but it is not guaranteed otherwise!

Important Exercise

→ Using this Remark, prove that the Moëbius bundle is not orientable

Hint: Remember it is a vector bundle on S^1 to describe the two charts

→ Proposition

Let M be a smooth manifold. The FAR

(1) M admits an orientation $\mathcal{O}$ (continuous)

(2) M admits an orientation signed Atlas $(\mathcal{A}, s)$

[i.e. $s : \{(U, \varphi)\} \longrightarrow \{\pm 1\}$

$\varphi \longrightarrow s(\varphi)$

. s satisfies $s(\psi) = \operatorname{sgn}(\det D_{\varphi(p)}(\psi \circ \varphi^{-1}))s(\varphi)$]

(1) ⟹ (2) if we have a orientation $\mathcal{O}$, then we can cover M with charts (U, φ) s.t. $\mathcal{O}_p\left(\frac{\partial}{\partial \varphi^i}\Big|_p\right) = \pm \operatorname{sgn}(D_p \varphi)$ is constant

⟹ this gives us $(\mathcal{A}, s)$

⟸ We want to define $\mathcal{O}_p(\varphi) = s(\varphi)$ [i.e. choose the orient. s.t. this is true]

such a define field is continuous ☑

§ 13.1 Volume Forms

Proposition [15.3 Lee ; 7.1.1 Notes]

M smooth n-dimensional manifold.

M is orientable $\iff$ $\exists\ \omega \in \Omega^n(M)$ nowhere vanishing

proof

($\Leftarrow$) We define $\mathcal{O}_p : T_pM \to \{\pm 1\}$ as

$$\mathcal{O}_p(v_1, \dots, v_n) = \operatorname{sgn} \omega_p(v_1, \dots, v_n).$$

We need to show that $\mathcal{O}$ defined in this way is continuos.

$\iff \forall p, \exists (U, \varphi)$ s.t. $\mathcal{O}_p\left(\frac{\partial}{\partial \varphi^i}\Big|_p\right)$ is constant in a neighb. of φ.

On such a open U, $\omega|_U = f\, d\varphi^1 \wedge \dots \wedge d\varphi^n$ with $f \in C^\infty(U)$, $f \neq 0$ on U

$$\Rightarrow \mathcal{O}_p\left(\frac{\partial}{\partial \varphi^i}\Big|_p\right) = \operatorname{sgn} f(p).$$

Up to shrinking U, we can assume it is connected

$\Rightarrow$ since f is smooth and never 0 is sign is constant ✓

($\Rightarrow$) If M was a continuous orientation $\mathcal{O}$

$\Rightarrow$ each point $\underline{\underline{p \in M}}$ $p \in (U, \varphi)$ s.t.

($\boxtimes$) $\mathcal{O}_p\left(\frac{\partial}{\partial \varphi^i}\Big|_p\right) =$ constant in $p \in U' \subseteq U$

$\Rightarrow$ We can choose a atlas $\mathcal{A} = \{(U', \varphi)\}$ s.t.

($\boxtimes$) is satisfied on all of U'

Then, on each (U', φ) we define

$$\omega_\varphi = \mathcal{O}(\varphi) = d\varphi^1 \wedge \_\_ d\varphi^n) .$$

Now, using partions of 1, we glue ω_φ to a global, nowhere vanishing n-form.

$\Rightarrow$ Choose $\{\eta_\varphi\}_{\varphi \in A}$ a POU subordinate to $\{(U', \varphi)\}$

$\Rightarrow$ Define $\boxed{\omega = \sum_{\varphi \in A} \eta_\varphi \, \omega_\varphi}$

$\to$ smooth because it is smooth on each by construction.

$\Rightarrow$ ω is nowhere vanishing :

$\forall p$, let $\langle b_1, \dots b_n \rangle = T_pM$

$$\Rightarrow \omega_p(b_1, \dots, b_n) = \sum_{\varphi \in A} \underbrace{\eta_\varphi(p)}_{\substack{\text{sum to } 1, \\ \geq 0}} \cdot \underbrace{\omega_\varphi|_p(b)}_{\underbrace{\operatorname{sgn}(\mathcal{O}_p(b))}_{\text{"const}}} \Rightarrow \neq 0 \quad \forall \varphi \quad \blacksquare$$

Corollary

Let $S \hookrightarrow (M, \mathcal{O})$ an embedded hypersurface.

A vector field Y on M is said transverse to S if $\forall p \in S$, $Y_p \in T_pM \setminus T_pS$.

Such a Y induces an orientation on S.

proof

on S we have $\omega \in \Omega^n(S)$ nowhere vanishing

$$Y \lrcorner \omega$$

$$\begin{tikzcd}
TM \arrow[rr, "\simeq", "{\langle di, Y\rangle}"'] \arrow[dr] & & TS \oplus \mathbb{R}_M \arrow[dl] \\
& M &
\end{tikzcd}$$

§13.2 Integration

Given $\omega \in \Omega^k(M)$ we define

$$\mathrm{supp}\,\omega = \overline{\{p \in M \mid \omega_p \neq 0\}}$$

Def 1, Let $(M, \mathcal{O})$ an oriented mfd, $\omega \in \Omega^n(M)$ s.t. $\mathrm{supp}\,\omega$ is compact and $\mathrm{supp}\,\omega \subseteq (U, \varphi)$ s.t. $\mathcal{O}_p(\varphi) = \text{const}$

$$\Rightarrow \boxed{\int_M \omega := \mathrm{sgn}_{\mathcal{O}}(\varphi) \int_U (\varphi^{-1})^* \omega = \mathrm{sgn}_{\mathcal{O}}(\varphi) \int_{\varphi(U)} h \, dx}$$

Proposition

The definition is well posed, i.e. the integral does not depend from the choice of chart.

proof

- Let (U, φ), (V, ψ) two charts. Let us call

$W = U \cap V$, and

$$\sigma : \psi \circ \varphi^{-1} : \varphi(W) \to \psi(W)$$

- we have that

$$(\varphi^{-1})^* \omega = h \, dx^1 \wedge \ldots \wedge dx^n$$

$$(\psi^{-1})^* \omega = g \, dy^1 \wedge \ldots \wedge dy^n$$

with $g = h(\sigma^{-1}(y)) \cdot \det D_y(\sigma^{-1})$

Let's compare

- $$\int_M \omega = O(\varphi) \int_{\varphi(W)} h(x) \, dx^1 \wedge \ldots \wedge dx^n =$$

$$= O(\varphi) \int_{\psi(W)} h(\sigma^{-1}(y)) \, |\det D_y(\sigma^{-1})| \, dy^1 \wedge \ldots \wedge dy^n =$$

$$= \underbrace{O(\varphi) \cdot \operatorname{sgn}(\det D_y(\sigma^{-1}))}_{= \operatorname{sgn}(\psi)} \int_{\psi(W)} g \, dy^1 \wedge \ldots \wedge y^n$$

$\blacksquare$

Definition in general:

$(M, \mathcal{O})$ oriented manifold.

$\omega \in \Omega^n(M)$ compactly supported.

- let $\{(U_i, \varphi_i)\}$ be an atlas for M s.t.

$\mathcal{O}$ is constant on each U_i

- Fix $\{\eta_i\}$ a partition of unity subordinate to U_i

$\Rightarrow$ We define

$$\boxed{\begin{aligned} &\int_M \omega := \sum_i \int_M \eta_i \omega \\ \text{where} \quad &\int_M \eta_i \omega = \mathcal{O}(\varphi_i) \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\omega \cdot \eta_i) \end{aligned}}$$

Proposition

The above definition is well posed, i.e does not depend from the choice of charts, nor from the choice of POU

proof

- let $\{(U_i, \varphi_i)\}$, $\{(V_j, \psi_j)\}$ two atlas with consistently signed charts.
- let $\{\chi_i\}$, $\{\eta_j\}$ partition of unity subordinate to $\{U_i\}$, $\{V_j\}$ respectively.
- By definition

$$\int_M \omega = \sum_j \int_M \eta_j \omega = \sum_j \int_M \eta_j \sum_i \chi_i \omega =$$

$$= \sum_{i,j} \int_M \eta_j \chi_i \omega$$

here this is non zero and admits two expressions on $V_j \cap U_i$,

but the proposition above tells us they coincide ∎

Proposition (Exercise)

Let $(M, \mathcal{O})$ oriented smooth mfd.

(1) $\int_M a\omega + b\eta = a \int_M \omega + b \int_M \eta$

(2) If $\operatorname{sgn}(\omega|_p) = \mathcal{O}_p(\partial_{x^i})$ at every point

$$\Rightarrow \int_M \omega \geq 0 \text{ and } = 0 \iff \omega = 0$$

(3) If $F: N \to M$ is a diffeomorphism of oriented manifold, with constant sign $\operatorname{sign}(f) = \pm 1$

$$\Rightarrow \int_N F^* \omega = \operatorname{sgn} f \int_M \omega$$

(4) Let $(M, \mathcal{O})$, $(-M, -\mathcal{O})$ denote the mfds with reversed orientation

$$\Rightarrow \int_{-M} \omega = -\int_M \omega$$

§13.3 Stokes Theorem

- Recall that a manifold with boundary M is a second countable, Hausdorff topological space with a ~~[illegible]~~ atlas $\mathcal{A}$ (U, φ) s.t. $\varphi : U \to \tilde{U} \underset{\text{open}}{\subseteq} \mathbb{H}^n$ is homeo

  + change of charts are diffeomorphism

  [Attention here to what smooth means for $F : \tilde{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$]

Definition: Let M be a smooth manifold with boundary, $\mathcal{O}$ a continuous orientation on M

$\Rightarrow$ we define the boundary orientation $\partial\mathcal{O}$ for ∂M as follows:

$$\partial\mathcal{O}_p(X_1, \ldots X_{n-1}) = \mathcal{O}_p(Y, X_1, \ldots X_n)$$

for $\langle X_1, \ldots X_{n-1} \rangle = T_p\partial M$

$Y \in T_pM$ an outward pointing

tangent vector: $T_pM \simeq \mathbb{R}^n$

$$v = \sum v^i \frac{\partial}{\partial x^i}\Big|_p$$

where the coordinates are with respect to a chart φ s.t. $\varphi|\partial$ is a chart

$v^n < 0 \Rightarrow$ outward pointing

Exercise:

- Show that the definition is well posed, i.e. independent from the choice of outward-pointing vectors
- Show that if $\mathcal{O}$ is continuous $\Rightarrow$ $\partial\mathcal{O}$ is continuous

Using the characterization in terms of volume forms, you have that

$$\partial\omega = \omega(Y_0, -)$$

Examples

① Let $(M, \mathcal{O})$ be $[a,b] \subseteq \mathbb{R}$ with orientation induced by the standard orientation

$\mathbb{H}^+ = \{ x \mid x \geq 0 \}$ 

$\Rightarrow$ an atlas for $[a,b)$ is given by:

$$\varphi_1 : [a,b) \longrightarrow [0, b-a)$$
$$x \longrightarrow x-a$$

$$\varphi_2 : [a,b] \longrightarrow [0, b-a)$$
$$x \longrightarrow b-x$$

$\Rightarrow Y_a = -\frac{\partial}{\partial x}\Big|_a \qquad Y_b = \frac{\partial}{\partial x}\Big|_b$

$\Rightarrow \partial\,\mathcal{O}(a) = -1$

$\partial\,\mathcal{O}(b) = +1$

② Let $S^2 = \partial B^3 = \{ x \in \mathbb{R}^3 \mid \|x\| \leq 1 \}$

And let consider on $(B^3, \mathcal{O}_{std})$

$\Leftrightarrow \mathcal{O}_\omega$ for $\omega = dx^1 \wedge dx^2 \wedge dx^3$

$\Rightarrow \partial\mathcal{O}_\omega$ is the one induced by

$\partial\omega_p(X,Y) = \omega(N_p, X, Y)$ $\quad N_p$ = outward pointing vector field

$N_p = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$

③ $M = \mathbb{H}^n, \mathcal{O}_{std} \quad \partial\mathbb{H}^n \simeq \mathbb{R}^{n-1}$

$(x^1, \ldots x^{n-1}, 0) \to (x^1, \_\, x^{n-1})$

$\Rightarrow Y$ outward pointing is given by $\boxed{-\frac{\partial}{\partial x^n}}$

$$\partial\mathcal{O}_{std} = \operatorname{sgn}\left((dx_1 \wedge \ldots \wedge dx^n \lrcorner \left(-\frac{\partial}{\partial x^n}\right) \right)$$

$$\| $$

$$\underbrace{(-1)}_{\text{sgn of vect. field}} \cdot \underbrace{(-1)^{n-1}}_{\text{Full is antisymmetric}} dx^1 \wedge \ldots \wedge dx^{n-1}$$

$\rightarrow$ in n even $\Rightarrow \partial\mathbb{H}^n$ has $\mathcal{O}^{n-1}_{std}$

else $\quad$ " $\quad -\mathcal{O}^{n-1}_{std}$

Let $\omega \in \Omega^k(M)$ for some $k \leq n$

$\Rightarrow \forall$ smooth map $F: N \to M$ such that :
- N is oriented k-manifold
- $F^*\omega$ has compact support

We can define

$$\int_N F^*\omega.$$

<u>Proposition</u> : $\omega, \theta \in \Omega^k(M)$

$\Rightarrow \omega = \theta$

$$\Leftrightarrow \int_N F^*\omega = \int_N F^*\eta \quad \forall \; N \xrightarrow{F} M$$

with N as before

<u>Theorem</u> : <u>Stoke's Theorem</u>

Let $\omega \in \Omega^{n}(M)$, M $n+1$-dimensional manifold with boundary, ω a compactly supported form

$$\Rightarrow \int_M d\omega = \int_{\partial M} i^*\omega$$

$i : \partial M \hookrightarrow M$