Introduction to Differentiable Manifolds

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Solutions Series 6 - Submanifolds (more!) **2021–11–08**

Exercise 6.1. Consider the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the projection map.

(a) Give $\mathbb{T}^n$ a natural smooth structure so that π is a local diffeomorphism.

Solution. We have already seen in a previous exercise that π is locally injective. This means that $\mathbb{R}^2$ is covered by open sets U such that the restriction $\pi|_U : U \to \mathbb{T}^n$ is injective. We take these maps $\phi = \pi|_U$ as local parametrizations of $\mathbb{T}^n$. Their inverses form a smooth atlas for $\mathbb{T}^n$. (The transition maps are locally translations, hence smooth.) □

(b) Show that a map $f : \mathbb{T}^n \to M$ (where M is a smooth manifold) is smooth if and only if the composite $f \circ \pi$ is smooth.

Solution. If f is $\mathcal{C}^k$, it is clear that $f \circ \pi$ is $\mathcal{C}^k$.

Now suppose $f \circ \pi$ is $\mathcal{C}^k$. To show that f is $\mathcal{C}^k$, it suffices to show that $f \circ \phi$ is $\mathcal{C}^k$ for all parametrizations $\phi = \pi|_U$ as above. And indeed, by decomposing $\phi = \pi \circ \iota_U$, where $\iota_U : U \to \mathbb{R}^n$ is the inclusion map, we see that the map $f \circ \phi$ is $\mathcal{C}^k$ because $f \circ \phi = f \circ \pi \circ \iota_U$ and both $f \circ \pi$ and ι_U are $\mathcal{C}^k$. □

(c) Show that $\mathbb{T}^n$ is diffeomorphic to the product of n copies of the circle $\mathbb{S}^1$.

Solution. Recall the homeomorpism $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1 \subseteq \mathbb{R}^2$ that sends $[t] \mapsto (\cos(2\pi t), \sin(2\pi t))$. We will construct an n-dimensional version of it.

For this exercise it is convenient to define the torus as $\mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$. We define a map $f : \mathbb{R}^n \to (\mathbb{S}^1)^n \subseteq \mathbb{R}^{2n}$ that sends

$$(t^i)_{0\leq i<n} \mapsto (\cos t^0, \sin t^0, \cos t^1, \sin t^1, \dots).$$

Since the map f is $2\pi\mathbb{Z}^n$-periodic, by the previous part of the exercise it passes to the quotient giving a smooth map $\overline{f} : \mathbb{T}^n \to (\mathbb{S}^1)^n$ that satisfies $f = \overline{f} \circ \pi$.

Note that map $\overline{f}$ is an immersion. To prove this, since π is a surjective, it suffices to check that the map $\iota \circ f = \iota \circ \overline{f} \circ \pi : \mathbb{R}^n \to \mathbb{R}^{2n}$ is an immersion, where ι is the inclusion map $(\mathbb{S}^1)^n \to \mathbb{R}^{2n}$. To see that $\iota \circ \overline{f}$ is an immersion we note that the n vectors

$$\mathrm{T}_p(\iota \circ f)(e_i) = (0, \dots, 0, -\sin t^i, \cos t^i, 0, \dots, 0)$$

are linearly independent, since they are nonzero and contained in different coordinate planes.

Since $\overline{f} : \mathbb{T}^n \to (\mathbb{S}^1)^n$ is an immersion between n-dimensional manifolds, it follows that $\overline{f}$ is a local diffeomorphism, and in particular it is an open map. Since in addition $\overline{f}$ is bijective, it is a diffeomorphism. □

Exercise 6.2. If S is an embedded submanifold of M, show that there is a unique topology and smooth structure on S such that the inclusion map $S \to M$ is an embedding.

Solution. Recall that gives $S \xrightarrow{j} M$ an embedding we have defined a smooth manifold structure on M as follows: we endow S with the subspace topology and we define smooth charts via the *Slice property for embedded submanifold* (Notes: Proposition 3.3.2, Lee's book Theorem 5.8). This means that smooth charts for S are given by taking $(U \cap S, \varphi|_U)$ for (U, φ) a smooth chart for M such that $\varphi(U \cap S) \subset \varphi(U)$ is given by $\{q \in \varphi(U) \mid x^{k+1}(q) = \dots = x^n(q) = 0\}$ where $k = \dim S$ and $n = \dim M$.

We want to show that this is the unique smooth structure on S for which j is indeed an embedding.

Recall that by the *initial property of embedded submanifolds* (Proposition 3.3.3+ Proposition 3.3.4 Notes; Theorem 3.29+Corollary 3.30 Lee's book), if $F\colon N \to M$ is a smooth map whose image is contained in S then $F\colon N \to S$ is smooth.

Suppose there exists some other topology and smooth structure on S making it into an embedded submanifold. Let us denote by $\widetilde{S} \xrightarrow{\tilde{j}} M$ the same subspace inclusion but endowed with the different structure. By definition of emdedding, $\tilde{j}$ is smooth. Moreover, we have that $\tilde{j}(\widetilde{S}) = S$ and thus by the initial property recalled above $\tilde{j}\colon \widetilde{S} \to S$ is also smooth. For each point p, consider

$$D_p\tilde{j}\colon T_p\widetilde{S} \to T_pM.$$

Since the image of $\widetilde{S}$ is contained in S we have a factorisation

$$T_p\widetilde{S} \xrightarrow{D_p\tilde{j}} T_pS \xrightarrow{D_pj} T_pM.$$

Since the composition is injective by definition of embedding, so is the first linear map. This means that $\tilde{j}\colon \widetilde{S} \to S$ is a smooth immersion. But a smooth immersion which is bijective is a diffeomorphism, so the smooth structure on $\widetilde{S}$ was the standard smooth structure on embedded submanifolds.

□

Exercise 6.3. For a subset S of a smooth manifold M, show that the following are equivalent:

(a) S is a closed embedded k-submanifold of M.
(b) For each point $p \in M$ there exists a chart (V, φ) that is k-sliced by S, i.e. we have

$$S \cap V = \{q \in V : \phi^k(q) = \cdots = \phi^{n-1}(q) = 0\}.$$

Solution. $(a) \Rightarrow (b)$ is the slice property for embedded submanifolds (Notes: Proposition 3.3.2, Lee's book Theorem 5.8).

Viceversa. First notice that since S is a subspace of a manifold it is automatically Haursdorff and second countable. We now define a smooth atlas on S as follow: let $\pi\colon \mathbb{R}^n_{x^0,\dots,x^{n-1}} \to \mathbb{R}^k_{x^0,\dots,x^{k-1}}$ the standard projection. Consider the composition:

$$S \cap V \xrightarrow{\phi} \phi(S \cap V) \subset \phi(V) \subset \mathbb{R}^n_{x^0,\dots,x^{n-1}} \xrightarrow{\pi} \mathbb{R}^k_{x^0,\dots,x^{k-1}}.$$

Remember that the following notation $\phi^k = x^k \circ \phi$ is being used.

Now notice that by the slice property $\phi(S \cap V)$ is a open inside $\pi^{-1}(\mathbb{R}^k)$. Moreover since the standard projection is open (see next solution for details) $\pi \circ \phi(S \cap V) \subseteq \mathbb{R}^k$ is open. Thus we have a continuos bijection into a open in $\mathbb{R}^k$

$$S \cap V \xrightarrow{\pi\circ\phi} U \subseteq \mathbb{R}^k.$$

This is in fact a homeomorphism since the inverse is given by $(\pi \circ \phi)^{-1}(x) = \phi^{-1}(x^0, \dots, x^{k-1}, 0, \dots, 0)$ which is continuos because composition of the standard immersion $j\colon \mathbb{R}^k \to \mathbb{R}^n$ and ϕ^{-1} which is continuos since it is a chart of M.

This prove that the the collection of $(V_S := V \cap S, \phi_S := \pi \circ \phi)$ for (V, ϕ) charts of a smooth atlas for M define a k-topological manifold structure on S.

To conclude we want to check that these charts are in fact smoothly compatible and so define a smooth structure on S. If (V_S, ϕ_S) and (V'_S, ϕ'_S) are two charts, the transition function is given by

$$\varphi'_S \circ (\varphi_S)^{-1} = \pi \circ \phi' \circ \phi^{-1} \circ j$$

which is smooth since composition of smooth maps. □

Exercise 6.4. Show that every submersion is an open map.

Solution. Let $F\colon M \to N$ a submersion. Let us start by argue that we can reduce the problem to proving that the standard projection $\mathbb{R}^n_{x^0,\dots,x^{n-1}} \xrightarrow{\pi} \mathbb{R}^k_{x^0,\dots,x^{k-1}}$ is open.

Let $W \subset M$ be a open subset and suppose we know the standard projection is open. By the costant rank theorem we can find charts $(U_i, \varphi_i), (V_i, \psi_i)$ covering M and N respectively such that

$$F^{\psi_i}_{\varphi_i}\colon \varphi_(U_i) \subseteq \mathbb{R}^m \to \psi_i(V_i) \subseteq \mathbb{R}^n$$

is the standard projection.

Then we have $F^{\psi_i}_{\varphi_i}(\varphi_i(W \cap U_i))$ is open for each i; in particular, since φ_i, ψ_i are homeomorphism onto their images and thus in particular open maps, $F(U_i \cap W) = \psi_i^{-1} \circ F^{\psi_i}_{\varphi_i}(\varphi_i(U_i \cap W))$ is open. Since $W = \bigcup W \cap U_i$ and thus $F(W) = \bigcup F(W \cap U_i)$ the latter is open because union of opens.

It remains to prove that the standard projection is open. Let us write $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$. We can choose as a base for the standard topology of $\mathbb{R}^n$ the family of opens of the form $U \times V$ for $U \subset \mathbb{R}^{n-k}$ and $V \subseteq \mathbb{R}^k$ are open sets. For opens in the basis the statement follows from the definition of the projection map. If $W \subseteq \mathbb{R}^n$ is any open set, then by definition of base for a topology this can be covered by opens of the form $U \times V$. This conclude the argument. ☐

Exercise 6.5. If M is a smooth manifold and $\pi : N \to M$ is a covering map, show that N has a unique smooth structure such that π is a local diffeomorphism.

Solution. Recall that a map of topological manifolds $\pi : N \to M$ is a covering map if it is continuos, surjective and every point p in M has a neighbourhood U such that every connected component V_j of $\pi^{-1}(U) = \bigsqcup V_j$ is mapped homeomorphically to U.

Fix a smooth atlas on M $\{U_i, \varphi_i\}$ such that each connected component V_ij of $\pi^{-1}(U_i) = \bigsqcup V_{ij}$ is mapped homeomorphically to U_i.

We can define a smooth atlas on N taking as charts $(V_{ij}, \psi_{ij} := \varphi_i \circ \pi_{ij})$ where π_{ij} simply denote the restriction of π to the open V_{ij}. These are homeomorphism into open subset of $\mathbb{R}^m$ by definition of covering map. If $(V_{ij}, \psi_{ij}), (V_{kl}, \psi_{kl})$ are two charts (notice that necessarily $i \neq k$ otherwise the intersection is empty and there is nothing to check) then

$$\psi_{ij} \circ \psi_{kl}^{-1} = \varphi_i|_{U_i \cap U_k} \circ \pi_{ij}|_{V_{ij} \cap V_{kl}} \circ (\pi_{kl}|_{V_{ij} \cap V_{kl}})^{-1} \circ (\varphi_k|_{U_i \cap U_k})^{-1} = \varphi_i|_{U_i \cap U_k} \circ (\varphi_k|_{U_i \cap U_k})^{-1}$$

which is smooth since $\{U_i, \varphi_i\}$ are a smooth atlas for M. With this smooth structure, $\pi : N \to M$ is a local diffeomorphism since, in local charts $\pi|^{\psi_{ij}}_{\varphi_i}$ is just the identity between two opens of $\mathbb{R}_n$

Let us argue that this is the unique smooth structure on N making π into a local diffeomorphism. Suppose there exist another smooth structure such that $\pi\colon \widetilde{N} \to M$ is a local diffeomorphism. Then there exists a smooth atlas $\{W_i, \phi_i\}$ of $\widetilde{N}$ such that $\pi\colon W_i \to \pi(W_i)$ is a diffeomorphism. Notice that since π is surjective and $\pi|_{W_i}$ is open, the $\pi(W_i)$ are a open cover of M. Up to shrinking the W_i we can assume that $\pi(W_i) \subseteq U_i$ and so $W_i \subseteq V_{ij}$ for some j. To conclude we need to show that ψ_i, ϕ_i are smoothly compatible. By assumption π is a local diffeomorphism with respect to these smooth structure, i.e: $\varphi_i \circ \pi \circ \phi_i^{-1}$ and $\varphi_i \circ \pi \circ \psi_i^{-1}$ are diffeomorphisms. But

$$\phi_i \circ \psi_i^{-1} = (\varphi_i \circ \pi \circ \phi_i^{-1}) \circ (\varphi_i \circ \pi \circ \psi_i^{-1})^{-1}$$

and the right hand side is clearly smooth since is composition of smooth maps. ☐

Exercise 6.6. Let $\iota : N \to M$ be a smooth embedding of smooth manifolds.

(a) If ι is a closed map, show that for every smooth function $f \in C^\infty(N)$ there exists a smooth function $g \in \mathcal{C}^\infty(M)$ such that $f = g \circ \iota$.
(b) Is this still true if we omit the assumption that ι is a closed map ?

Solution. (a) In Lecture 3 we proved, using partitions of unity, the following Extension Lemma:

Lemma: Let M be a smooth manifold, $A \subseteq M$ a closed subset, and $f\colon A \to \mathbb{R}$ a smooth function[1] then there exists a smooth function $\tilde{f}\colon M \to \mathbb{R}$ such that $\tilde{f}|_A = f$.

The idea was to define $\tilde{f} = \sum_{p\in A} \eta_p(x)\tilde{f}_p(x)$ for $\{\eta_p(x)\}$ a partition of unit subordinate to the open cover of M given by the opens $\{W_p, M \setminus A\}$.

To solve (a) we want to prove that given $\iota : N \to M$ a closed embedded submanifold and $f\colon N \to \mathbb{R}$ a smooth function with respect the unique smooth structure on N such that ι is an embedding, then f is also smooth in the sense we just recalled.

So we have to show that we can define for each $p \in M$ a open neighbourhood W_p of p in M and a smooth function $\tilde{f}_p\colon W_p \to \mathbb{R}$ such that $\tilde{f}_p|_{N\cap W_p} = f|_{N\cap W_p}$. Once this is done, we can apply the Extension Lemma.

The idea is once again to use the slice chart Lemma. Since $\iota\colon N \to M$ is an embedding, for each $p \in N$ there exists a coordinate chart (U, ϕ) of M centered at p such that

$$\phi(U \cap N) \xrightarrow{\phi\iota|_{N\cap U}} \phi(U)$$

is the standard embedding $\mathbb{R}^n \to \mathbb{R}^m$ defined by $(x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, 0 \dots, 0)$. In particular we can define smooth retractions $U \xrightarrow{\pi} U \cap N$ (i.e. $\pi \circ \iota = \mathrm{id}_{U\cap N}$) where $\pi := (\phi|_{U\cap N})^{-1} \circ \pi_n \circ \phi$ with $\pi_n\colon \mathbb{R}^m \to \mathbb{R}^n$ the standard projection $(x^1, \dots, x^m) \mapsto (x^1, \dots, x^n)$.

Then we define $W_p := U$ and

$$\widetilde{f_p} = f|_{U\cap N} \circ \pi.$$

The extension is smooth because composition of smooth function and clearly restrict to f on $N \cap U$ since $\pi \circ \iota = \mathrm{id}_{U\cap N}$.

(b) If the embedding is not closed, we can still extend the function to a open $N \subseteq V \subseteq M$ using the slice chart lemma and taking $V = \bigcup W_p$, but not necessarily to all of M. In fact, if N is itself open it might well be that it is already the biggest domain in which the function f is smooth. (See Exercise 1 in Problem sheet 3)

□

Exercise 6.7 (To hand in). (a) Show that the map $f : \mathbb{P}^2 \to \mathbb{R}^3$ defined by

$$f([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy).$$

is smooth, and has injective differential except at 6 points.

(b) Show that the map $g : \mathbb{P}^2 \to \mathbb{R}^4$ defined by

$$g([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy, x^2 - z^2)$$

is a smooth embedding.

Exercise 6.8. *

(a) A smooth map $f : M \to N$ is **transverse** to a closed embedded submanifold $S \subseteq N$ if for all points $p \in f^{-1}(S)$ we have $\mathrm{T}_{f(p)}S + \mathrm{Img}(\mathrm{T}_p f) = \mathrm{T}_{f(p)}N$. If this happens, show that $f^{-1}(S)$ is a closed embedded submanifold of M. What is its dimension ? What is its tangent space ?

[1] Let us recall that being A just a subset in this statement f smooth means that for each $p \in A$ there exists a open neighbourhood W_p of p in M and a smooth function $\tilde{f}_p\colon W_p \to \mathbb{R}$ such that $\tilde{f}_p|_{A\cap W_p} = f|_{A\cap W_p}$

(b) Two smooth maps $f_0 : M_0 \to N$ and $f_1 : M_1 \to N$ are **transverse** to each other if for any pair of points $p_0 \in M_0$, $p_1 \in M_1$ such that $f_0(p_0) = f_1(p_1) =: q \in N$ we have $\mathrm{Img}(\mathrm{T}_{p_0} f_0) + \mathrm{Img}(\mathrm{T}_{p_1} f_1) = \mathrm{T}_q N$. If this happens, prove that the set

$$S := \{(p_0, p_1) \in M_0 \times M_1 \mid f_0(p_0) = f_1(p_1)\}$$

is a closed submanifold of $M_0 \times M_1$. What is its dimension ?

Solution. (a) This is Theorem 6.30 in Lee's book (pag 144). We sketch the argument.

Since S is a submanifold, given any $x \in f^{-1}(S)$ we can find a chart (U, φ) for N around $f(x)$ such that $(S \cap U) = \{p \in U \mid \varphi^{k+1}(p) = \cdots = \varphi^n(p) = 0\}$, i.e. there exist a smooth fuction $g = \pi \circ \varphi \colon U \to \mathbb{R}^n \to \mathbb{R}^{n-k}$ such that $(S \cap U) = g^{-1}(0)$. Let us consider $g \circ f|_{f^{-1}(U)} \colon f^{-1}(U) \to \mathbb{R}^{n-k}$. Then $f^{-1}(S) \cap f^{-1}(U)$ is given by $(g \circ f)^{-1}(0)$, so to conclude it is enough to prove that 0 is a *regular value* for $(g \circ f)$ and the result follows from the regular Level set Theorem we proved in Lecture 5 (Theorem 3.5.2 Notes/ Theorem 5.12 Lee's book).

The value 0 is regular for g since it is the composition of a diffeomorphism and the standard projection, so given $z \in T_0\mathbb{R}^{n-k}$ exist $v \in T_f(x)N$ such that $D_f(x)g(v) = z$. The assumption of transversality tell us that we can write $v = v_0 + D_x f(w)$ for $v_0 \in T_f(x)S$ and $w \in T_x M$.

But since $g|_{(S \cap U)}$ is constant $D_f(x)g(v_0) = 0$ so $z = D_x(g \circ f)(w)$.

The argument also show that the codimension of $f^{-1}(S)$ in M is equal to the codimension of S is N.

Finally, by Exercise 5.2 the tanget space

$$T_x f^{-1}(S) = \mathrm{Ker}(T_x f^{-1}(U) \xrightarrow{g \circ f} T_0\mathbb{R}^{n-k})$$

(b) Consider the smooth map $F \colon M_0 \times M_1 \xrightarrow{(f_0, f_1)} N \times N$. There is a smooth embedding $N \hookrightarrow N \times N$ defined by $q \mapsto (q, q)$. Then $S = F^{-1}(N)$ and the transversality of f_0 and f_1 translates into the transversaliy of F and N in the sense of point (a).

□