Exercise 7.1. Show that the Moëbious bundle as defined in the lecture is a smooth vector bundle on S^1 (Exhibit local trivialization and compute the transition functions.)

Solution. The Mobius bundle is defined by $M := \mathbb{R} \times \mathbb{R}/\sim \cong M := [0,1] \times \mathbb{R}/\sim$ for $(x, y) \sim (x + n, -y)$ with $n \in \mathbb{Z}$

First we show that M is a smooth manifold. M is second countable because quotient of second countable, it is Hausdorff because quotient of a Hausdorff space by a discrete action. Define charts φ_1, φ_2 by restricting $\pi \colon \mathbb{R}^2 \to M$ to opens on which the quotient map is injective; e.g $V_1 = [0, 1)$, $V_2 = (0, 1]$, The transition function of the two charts,

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(V_1 \cap V_2) \to \varphi_2(V_1 \cap V_2),$$

is given by

$$\varphi_2 \circ \varphi_1^{-1}(x,t) = \begin{cases} (x,t) & x \in (0, \frac{1}{2}) \\ (x-1, -t) & x \in (\frac{1}{2}, 1) \end{cases}$$

So the two charts are smoothly compatible. We conclude M is a smooth manifold.

We show that M is a smooth vector bundle. Let $\pi : M \to S^1$, $\pi([(x,t)]) := [x]$.

Then, using the notation from the Example, over $U_1 \subset S^1$ we have the local trivialization

$$\Phi_1 : \pi^{-1}(U_1) \to U_1 \times \mathbb{R} : [(x,t)] \mapsto ([x], t),$$

which is a diffeomorphism, because it is the composition $(\nu_1^{-1} \times id_{\mathbb{R}}) \circ \varphi_1$ and both φ_1 and ν_1 are (trivially) diffeomorphisms, since they are smooth charts. (Note $\pi^{-1}(U_i) = V_i$.) Clearly Φ_1 is also a linear map on each fibre if we define the vector space structure on .

Similarly, over U_2 we define a local trivialization

$$\Phi_2 : \pi^{-1}(U_2) \to U_2 \times \mathbb{R} : [(x,t)] \mapsto \begin{cases} ([x], t) & x \in [0, \frac{1}{2}) \\ ([x], -t) & x \in (\frac{1}{2}, 1] \end{cases},$$

which is a diffeomorphism, because it is the composition $(\nu_2^{-1} \times id_{\mathbb{R}}) \circ \varphi_2$. Moreover, Φ_2 is linear on each fibre. □

Exercise 7.2. Show that the tangent bundle TS^1 is trivial.

Solution. Since $\mathbb{S}^1$ is diffeomorphic to $\mathbb{T}^1$, it suffices to show that the tangent bundle of the n-torus $\mathbb{T}^n$ is trivial.

We denote $\kappa : \mathbb{R}^n \to \mathbb{T}^n$ the quotient map, since the letter π is now used for the projection $\pi : \mathrm{T}\mathbb{T}^n \to \mathbb{T}^n$.

Recall that there is an inverse atlas of $\mathbb{T}^n$ consisting of the parametrizations $\phi = \kappa|_{\widetilde{U}} : \widetilde{U} \to U \subseteq \mathbb{T}^n$, where $\widetilde{U} \subseteq \mathbb{R}^n$ is any open set where κ is injective and $U = \kappa(\widetilde{U})$.

Each such parametrization ϕ of $\mathbb{T}^n$ induces a parametrization $\Phi : \widetilde{U} \times \mathbb{R}^n \to \pi^{-1}U$ of $\mathrm{T}\mathbb{T}^n$ that sends $(x, v) \mapsto (\phi_{\widetilde{U}}(x), \sum_i v^i \frac{\partial}{\partial(\phi^{-1})^i}\Big|_p)$. Note here that ϕ^{-1} is a chart of $\mathbb{T}^n$. The parametrizations Φ of this kind form an atlas of $\mathrm{T}\mathbb{T}^n$, which defines the smooth structure on $\mathrm{T}\mathbb{T}^n$.

We define a frame of $\mathrm{T}\mathbb{T}^n$ consisting of n vector fields E^i defined as follows. For each parametrization $\phi : \widetilde{U} \to U$ as above, we let

$$E^i(p) = \Phi(\phi^{-1}(p), e_i) \quad \text{for all } p \in U.$$

This formula defines $E^i|_U$. Let us check that E^i is well defined (i.e. that the formula agrees on an intersection $U \cap V$ of images of two parametrizations $\phi : \widetilde{U} \to U$, $\psi : \widetilde{V} \to V$. For this, recall that the transition map $\psi^{-1} \circ \phi$ between the parametrizations ϕ, ψ of $\mathbb{T}^n$ is locally a translation. Therefore the transition map between the parametrizations Φ, Ψ of $\mathrm{T}\mathbb{T}^n$ is

$$\Psi^{-1} \circ \Phi(x, v) = (\psi^{-1} \circ \phi(x), \mathrm{D}_{\phi(x)}(\psi^{-1} \circ \phi)(v)) = (\psi^{-1} \circ \phi(x), v)$$

since the differential of a translation is the identity map. Equivalently, we have $\Phi(x, v) = \Psi(y, v)$ if $\phi(x) = \psi(y)$. In particular, for a point $x \in U \cap V$, putting $x = \phi^{-1}(p)$ and $y = \psi^{-1}(p)$, we have

$$\Phi(\phi^{-1}(p), e_i) = \Phi(x, y) = \Psi(y, e_i) = \Psi(\psi^{-1}(p), e_i),$$

as needed to show that E_i is well defined.

The vector fields E^i are clearly smooth because they are smooth on each open set U as above, since the maps Φ and ϕ^{-1} are smooth. The vector fields E^i are also linearly independent at each point $p = \phi(x) \in \mathbb{T}^n$, since the vectors e_i are linearly independent. Therefore the vectors E_i constitute a frame of $\mathrm{T}TT^n$, defined globally (i.e. on the whole torus $\mathbb{T}^n$). We conclude that that the tangent bundle $\mathrm{T}\mathbb{T}^n$ is trivial.

□

Exercise 7.3 (Properties of smooth vector fields)**.** Let M be a smooth manifold and let $X : M \to TM$ be a vector field. Show that the following are equivalent:

(a) X is a smooth vector field.

(b) The component functions of X are smooth with respect to all charts of one particular smooth atlas of M.

(c) For any smooth function $f : U \to \mathbb{R}$ on an open set $U \subset M$, the function $Xf : U \to \mathbb{R}$ defined by $Xf(p) := X_p(f)$ is smooth.

Solution. Let $(M, \mathcal{A})$ be a smooth manifold and X a vector field. Recall that we say that X is a smooth vector field if the component functions of X are smooth for any chart $(U, \varphi) \in \mathcal{A}$. The component functions w.r.t (U, φ) were defined as the functions $X^i : U \to \mathbb{R}$ such that

$$X_p = \sum_i X^i(p) \left.\frac{\partial}{\partial \varphi^i}\right|_p, \quad p \in U.$$

$(a) \Rightarrow (b)$ is clear.

$(b) \Rightarrow (a)$ Let $\mathcal{A}' \subset \mathcal{A}$ and suppose the component functions are smooth wrt all $(U, \varphi) \in \mathcal{A}'$. Let $(V, \psi) \in \mathcal{A}$. We write

$$X_p = \sum_i \widetilde{X}^i(p) \left.\frac{\partial}{\partial \psi^i}\right|_p, \quad p \in V$$

where $\widetilde{X}^i$ are the component functions of X wrt (V, ψ). To show that the $\widetilde{X}^i$ are smooth on V it suffices to show that they are smooth in a neighborhood of every point of V. So let $p \in V$, let $(U, \varphi) \in \mathcal{A}'$ be a chart containing p and let X^i be the component functions of X wrt (U, φ). Then from the change of coordinates formula it follows that (Exercise 3.iii from last week)

$$\widetilde{X}^i(q) = \sum_j \left(\left.\frac{\partial}{\partial \varphi^j}\right|_q \psi^i \right) X^j(q), \quad q \in U \cap V$$

and we conclude that $\widetilde{X}^i$ is smooth on $U \cap V$, $i = 1, \ldots, n$.

$(c) \Rightarrow (a)$ Let $(U, \varphi) \in \mathcal{A}$. Applying X to one of the components of φ yields $X\varphi^i = X^i$, which is smooth by hypothesis, i.e. the component functions of X wrt (U, φ) are smooth.

$(a) \Rightarrow (c)$ Conversely, suppose X is a smooth vector field, let $f \in \mathcal{C}^\infty(U)$ for an open set $U \subset M$. To check that Xf is smooth, it suffices to check that it is smooth in a neighborhood of every point of U. Given $p \in U$, let (W, φ) be a smooth chart containing p and satisfying $W \subset U$. Then on W we can write

$$Xf(q) = \sum_i X^i(q) \left.\frac{\partial}{\partial \varphi^i}\right|_q f$$

Then Xf is smooth on W since the component function of X are smooth by hypothesis and f is smooth (so in particular $\left.\frac{\partial}{\partial \varphi^i}\right|_q f = \left.\frac{\partial(f\circ\varphi^{-1})}{\partial x^i}\right|_{\varphi(q)}$ is smooth as a function of $q \in W$). □

Exercise 7.4. (**To hand in**) Show that there is a smooth vector field on S^2 which vanishes at exactly one point.

Hint: Try using stereographic projection and consider one of the coordinate vector fields.

Exercise 7.5 (Transition functions and vector bundles)**.** (a) Let $E \xrightarrow{\pi} M$ be a smooth vector bundle of rank k. Suppose that $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M; and for each α we are given a smooth local trivialization

$$\Phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k.$$

For each α, β, let

$$\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$$

[1] be the transition function defined by $\Phi_\alpha \circ \Phi_\beta^{-1}$. Show that the following identity is satisfied for all α, β, γ

$$\tau_{\alpha\beta} \circ \tau_{\beta\gamma} = \tau_{\alpha\gamma}$$

This is called the *cocycle condition* (The juxtaposition on the left-hand side represents matrix multiplication.)

(b) Suppose that $\{U_\alpha\}_{\alpha\in A}$ is an open cover of a smooth manifold M and that for each α, β, we are given smooth maps $\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ satisfying the identity above. Then there exists a smooth vector bundle $E \xrightarrow{\pi} M$ with local trivializations $\Phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ and transition functions $\tau_{\alpha\beta}$

Solution. (a) Let us denote by $V = U_\alpha \cap U_\beta \cap U_\gamma$. Then simply using the definition $\tau_{\alpha\beta} = \Phi_\alpha|_{U_\alpha\cap U_\beta} \circ (\Phi_\beta|_{U_\alpha\cap U_\beta})^{-1}$ we get

$$\tau_{\alpha\beta}|_V \circ \tau_{\beta\gamma}|_V = \Phi_\alpha|_V \circ (\Phi_\beta|_V)^{-1} \circ \Phi_\beta|_V \circ (\Phi_\gamma|_V)^{-1} = \tau_{\alpha\gamma}|_V$$

(b) Let us define $E = \bigsqcup_\alpha \{\alpha\} \times U_\alpha \times \mathbb{R}^k / \sim$ where

$$(\alpha, p, v) \cong (\beta, q, w) \Leftrightarrow q = p,\ w = \tau_{\alpha\beta}(p)(v)$$

Since $\tau_{\alpha\beta}$ satisfy the cocycle condition, this is an equivalence relation. $\pi \colon E \to$ is defined by $\pi([(\alpha, p, v)]) = p$. Then $\Phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ defined by $\Phi_\alpha([(\alpha, p, v)]) = (p, v)$ is bijective and the restriction to a point gives a linear isomorphism $\pi^{-1}(p) = \left\{[(\alpha, p, v)] | v \in \mathbb{R}^k\right\} \to \mathbb{R}^k$. Finally, if $U_\alpha \cap U_\beta \neq \emptyset$ the transition function

$$\Phi_\alpha \circ \Phi_\beta^- 1 \colon U_\alpha \cap U_\beta \times \mathbb{R}^k \to U_\alpha \cap U_\beta \times \mathbb{R}^k$$

is given by $(p, v) \to (p, \tau_{\alpha\beta}(p)(v))$. By the Vector bundle chart Lemma (Lemma 10.6 Lee's book), E is a vector bundle on M.

Let us say again in words how one gets E: for any vector bundle $E \xrightarrow{\pi} M$ there exist a *trivializing cover*, i.e. a cover $\{U_\alpha\}_{\alpha\in A}$ such that

$$E|_{U_\alpha} = \pi^{-1}(U_\alpha) \xrightarrow{\Phi_\alpha} U_\alpha \times \mathbb{R}^k$$

[1] for each point p $\tau_{\alpha\beta}(p)$ is a matrix

whit Φ_α a diffeomorphism. The transition functions $\tau_{\alpha\beta}$ tell us how these trivial vector bundle are glued along the intersections (the vector bundle is trivial on all of M if all the $\tau_{\alpha\beta}(p) = Id_{E_p}$. The cocycle condition is telling us that the gluing has to be consistent on triple intersections.

This exercise is showing how to built up E from the local trivialization and the gluing data.

□