Exercise 8.1. We have seen that given $X \in \mathfrak{X}(M)$ a vector field we have a an $\mathbb{R}$-*linear derivation*

$$X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

defined by

$$X(f) \colon M \to \mathbb{R},\ X(f)(p) = X_p f$$

where $X_p \in T_pM$ is the value of the vector field at p and $X_p f$ is given as described in Lecture 3.

(a) Show that the Lie bracket $[X,Y]$ defined by $[X,Y]f = X(Y(f)) - Y(X(f))$ is still a $\mathbb{R}$-linear derivation $[X,Y] \colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$.

(b) Suppose that we have coordinate expressions $X = \sum_i X^i \frac{\partial}{\partial x^i}$, $Y = \sum_j Y^j \frac{\partial}{\partial x^j}$. Prove that the Lie bracket is given in coordinates by

$$[X,Y] = \sum_i \sum_j (X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}) \frac{\partial}{\partial x^j}$$

(This exercise is extremely painful, I know! But you do it once in your life and never again.. As ugly as they are, vector fields and their brackets is what allow us to talk about direction derivatives on a manifold and to understand when two direction derivatives commute and when they don't! Notice that the Lie bracket of two coordinatye vector fields in $\mathbb{R}^n$ is always 0)

Solution. (a) To show that $[X,Y]$ is a $\mathbb{R}$-linear derivation, we need to check that it is indeed $\mathbb{R}$-linear and it satisfies the product rule, that is

$$[X,Y](\lambda f + g) = \lambda [X,Y](f) + [X,Y](g)$$

and

$$[X,Y](fg) = f[X,Y](g) + g[X,Y](f)$$

The linearity is straightful:

$$\begin{aligned}
[X,Y](\lambda f + g) &= (XY - YX)(\lambda f + g) = XY(\lambda f + g) - YX(\lambda f + g) \\
&= X(\lambda Y(f) + Y(g)) - Y(\lambda X(f) + X(g)) \\
&= \lambda XY(f) + XY(g) - \lambda YX(f) + YX(g) \\
&= \lambda (XY - YX)(f) + (XY - YX)(g) = \lambda [X,Y](f) + [X,Y](g)
\end{aligned}$$

where we used the linearity of X and Y.
For the product rule we have

$$\begin{aligned}
[X,Y](fg) &= (XY - YX)(fg) = XY(fg) - YX(fg) \\
&= X(fY(g) + gY(f)) - Y(fX(g) + gX(f)) \\
&= X(fY(g)) + X(gY(f)) - Y(fX(g)) - Y(gX(f)) \\
&= X(f)Y(g) + fXY(g) + X(g)Y(f) + gXY(f) \\
&\quad - Y(f)X(g) - fYX(g) - Y(g)X(f) - gYX(f) \\
&= f(XY - YX)(g) + g(XY - YX)(f) \\
&= f[X,Y](g) + g[X,Y](f)
\end{aligned}$$

hence the Lie bracket of two vector field is a derivation (a vector field).

(b) Let $X = \sum_i X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_j X^j \frac{\partial}{\partial x^j}$ be coordinate expressions of our vector field in a coordinate system $(U, \varphi = (x^1, \cdots, x^n))$ for M. Let us compute the vector field $[X, Y]$ in coordinates.

$$
\begin{aligned}
[X,Y](f) =& \\
&= XY(f) - YX(f) \\
&= (\sum_i X^i \frac{\partial}{\partial x^i})(\sum_j Y^j \frac{\partial}{\partial x^j})(f) - (\sum_j Y^j \frac{\partial}{\partial x^j})(\sum_i X^i \frac{\partial}{\partial x^i})(f) \\
&= (\sum_i X^i \frac{\partial}{\partial x^i})(\sum_j Y^j \frac{\partial}{\partial x^j}(f)) - (\sum_j Y^j \frac{\partial}{\partial x^j})(\sum_i X^i \frac{\partial}{\partial x^i}(f)) \\
&= \sum_{i,j} X^i \frac{\partial}{\partial x^i}(Y^j \frac{\partial}{\partial x^j}(f)) - \sum_{i,j} Y^j \frac{\partial}{\partial x^j}(X^i \frac{\partial}{\partial x^i}(f)) \\
&= \sum_{i,j}(X^i Y^j \frac{\partial^2}{\partial x^i \partial x^j}(f) + X^i \frac{\partial}{\partial x^i}(Y^j)\frac{\partial}{\partial x^j}(f)) \\
&\quad - \sum_{i,j}(X^i Y^j \frac{\partial^2}{\partial x^i \partial x^j}(f) + Y^j \frac{\partial}{\partial x^j}(X^i)\frac{\partial}{\partial x^i}(f)) \\
&= \sum_{i,j} X^i \frac{\partial}{\partial x^i}(Y^j)\frac{\partial}{\partial x^j}(f) - \sum_{i,j} Y^j \frac{\partial}{\partial x^j}(X^i)\frac{\partial}{\partial x^i}(f) \\
&= \sum_{i,j} X^i \frac{\partial}{\partial x^i}(Y^j)\frac{\partial}{\partial x^j}(f) - \sum_{j,i} Y^i \frac{\partial}{\partial x^i}(X^j)\frac{\partial}{\partial x^j}(f) \\
&= \sum_{i,j}(X^i \frac{\partial}{\partial x^i}(Y^j) - Y^i \frac{\partial}{\partial x^i}(X^j))\frac{\partial}{\partial x^j}(f)
\end{aligned}
$$

which is what we want.
Where, in the fifth equality, we used the linearity and in the sixth the product rule. Finally in the 8th equality we just changed i with j in the second term and then regrouped in one big sum.

□

Exercise 8.2. Let $f : M \to N$ be a smooth map. A vector field $X \in \mathfrak{X}(M)$ is **f-related** to a vector field $Y \in \mathfrak{X}(N)$ if $D_p f(X_p) = Y_{f(p)}$ for all $p \in M$.

(a) X is f-related to Y if and only if $X_p(h \circ f) = Y_{f(p)}(h)$ for all functions $h \in \mathcal{C}^\infty(N, \mathbb{R})$ and all points $p \in M$.

Solution. By definition of $D_p f$ we have $(D_p f(X_p))(h) = X_p(h \circ f)$ for all functions $h \in \mathcal{C}^\infty(M)$. Thus

$$
\begin{aligned}
X \text{ is } f\text{-related to } Y \text{ at } p &\iff Y_{f(p)} = D_p f(X_p) \\
&\iff Y_{f(p)}(h) = (D_p f(X_p))(h) \quad \forall h \in \mathcal{C}^\infty(N) \\
&\iff Y_{f(p)}(h) = X_p(h \circ f) \quad \forall h \in \mathcal{C}^\infty(N)
\end{aligned}
$$

□

(b) If X is f-related to Y and γ is an integral curve of X, show that $f \circ \gamma$ is an integral curve of Y.

Solution. We just need to verify that

$$(f \circ \gamma)'(t) = \mathrm{T}_{\gamma(t)} f(\gamma'(t)) = \mathrm{T}_{\gamma(t)} f(X_{\gamma(t)}) = Y_{f(\gamma(t))} = Y_{f \circ \gamma(t)}$$

for all t in the domain of γ. □

(c) If f is a local diffeo, for every vector field $Y \in \mathfrak{X}(N)$ there exists a unique $X \in \mathfrak{X}(M)$ that is f-related to Y. We denote $f^*Y := X$.

Thus if f is a diffeo, f-relatedness is a bijection from $\mathfrak{X}(M)$ to $\mathfrak{X}(N)$. In this case, if X is f-related to Y, we write $X = f^*Y$ and $Y = f_*X$.

Solution. Assume $f : M \to N$ is a local diffeo. Thus for every point $p \in M$, the linear transformation $D_pf : M \to N$ is invertible.

Let $Y \in \mathfrak{X}(N)$. A vector field X on M is f-related to Y iff for each point $p \in M$ we have $D_pf(X|_p) = Y_p$, or, equivalently, $X|_p = (D_pf)^{-1}(Y_{f(p)})$. Thus there is a unique vector field that is f-related to Y, and it is the function $p \mapsto (D_pf)^{-1}(Y_{f(p)})$. □

(d) If f is a closed embedding, show that every vector field $X \in \mathfrak{X}(M)$ is f-related to some vector field $Y \in \mathfrak{X}(N)$.

Hint: Construct Y locally, then use partitions of unity.

What happens if f is just an immersion? In this case, find and prove a local version of the fact.

Solution. The local version is the following.

Lemma. Let $f : M \to N$ be a smooth immersion, and let $X \in \mathfrak{X}(M)$. Then for each point $p_0 \in M$ there exist open neighborhoods $U \subseteq M$ and $V \subseteq N$ of p_0 and $f(p_0)$ resp., and a vector field $Y \in \mathfrak{X}(V)$ such that $X|_U$ is f-related to Y.

Proof. By the constant rank theorem, there exist charts $\varphi : U \to \widetilde{U}$ and $\psi : V \to \widetilde{V}$ of M and N centered at p_0 and $f(p_0)$ such that the local expression $\widetilde{f} = \psi \circ f \circ \varphi^{-1}$ of f is given by $\widetilde{f}(x^0, \dots, x^{m-1}) = (x^0, \dots, x^{m-1}, 0, \dots, 0)$. Moreover, we can assume that $\widetilde{V} = \widetilde{U} \times \widetilde{W}$ for some open set $\widetilde{W} \subseteq \mathbb{R}^{n-m}$ that contains the origin.

Note that each coordinate vector field $\frac{\partial}{\partial \varphi^i} \in \mathfrak{X}(U)$ of the chart φ is f-related to the corresponding coordinate vector field $\frac{\partial}{\partial \psi^i} \in \mathfrak{X}(V)$ of the chart ψ. That is, for each $p \in U$ we have $D_pf\left(\left.\frac{\partial}{\partial \varphi^i}\right|_p\right) = \left.\frac{\partial}{\partial \psi^i}\right|_{f(p)}$.

Let $\pi : \widetilde{V} \to \widetilde{U}$ be the projection map $(x^0, \dots, x^{m-1}) \mapsto (x^0, \dots, x^{n-1})$, and let $\rho = \varphi^{-1} \circ \pi \circ \psi : U \to W$. Note that ρ is a retraction of $f|_U^V$.

Let X^i be the components of X w.r.t. the chart φ. Thus $X|_U = \sum_{0 \le i < n} X^i \frac{\partial}{\partial \varphi^i}$ We construct a vector field $Y \in \mathfrak{X}(V)$ whose components w.r.t. the chart ψ are

$$Y^i = \begin{cases} X^i \circ \rho & \text{if } i < n, \\ 0 & \text{if } i \ge n. \end{cases}$$

Thus $Y|_q = \sum_{0 \le i < n} X^i(\rho(q)) \left.\frac{\partial}{\partial \psi^i}\right|_q$ for each point $q \in V$.

In particular, at a point $q = f(p)$ we have $\rho(q) = p$, therefore

$$\begin{aligned} Y|_q &= \sum_{0 \le i < n} X^i(p) \left.\frac{\partial}{\partial \psi^i}\right|_q \\ &= \sum_{0 \le i < n} X^i(p) D_pf\left(\left.\frac{\partial}{\partial \varphi^i}\right|_p\right) \\ &= D_pf\left(\sum_{0 \le i < n} X^i(p) \left.\frac{\partial}{\partial \varphi^i}\right|_p\right) = D_pf(X_p). \end{aligned}$$

This shows that Y is $f|_U^V$-related to X. □

Now we can prove the global version.

Let $f : M \to N$ be a closed embedding, and let $X \in \mathfrak{X}(M)$. We shall construct a vector field $Y \in \mathfrak{X}(N)$ such that X is f-related to Y.

The closed set $f(M)$ can be covered by open sets $(V_k)_{k\geq 1}$ where there is a vector field $Y_k \in \mathfrak{X}(V_k)$ that is f-related to X.

We also define the open set $V_0 = N \setminus f(M)$ and we put any vector field Y_0 on V_0, for example $Y_0 \equiv 0$. Note that X is f-related to Y_0 trivially. The open sets $(V_k)_{k\geq 0}$ form an open cover of N. Let $(\eta_k)_{k\geq 0}$ be a partition of unity subordinate to this cover, and consider the vector field $Y = \sum_k \eta_k Y_k \in \mathfrak{X}(N)$. We claim that X is f-related to Y. Indeed, for each point $p \in M$ we have

$$Y_{f(p)} = \sum_k \eta_k(f(p))\, Y_k|_{f(p)} = \sum_k \eta_k(f(p))\, D_p f(X|_p) = D_p f(X|_p)$$

because $\sum_k \eta_k(f(p)) = 1$. □

Solution. Let $\iota : S \to M$ be the inclusion map, and let $Y = X|_S \in \mathfrak{X}(S)$. Note that Y is ι-related to X.

Let $\gamma : I \to M$ be an integral curve of X that visits S. The set $I' = \gamma^{-1}(S) = \{t \in I : \gamma(t) \in S\}$ is nonempty and closed (because S is closed). We want to prove that $I' = I$, and for this it suffices to show that I' is open. Let $t_0 \in I'$. This means that $\gamma(t_0) \in S$. Let $\beta : J \to S$ be an integral curve of Y that coincides with γ at the instant t_0, where $J \subseteq \mathbb{R}$ is an open interval containing t_0. Since Y is ι related to X, the curve $\iota \circ \beta$ is an integral curve of X that coincides with γ at t_0, thus it coincides with γ in the interval $I'' = I \cap J$, which is a neighborhood of t_0. This implies that $\gamma(t) \in S$ for all $t \in I''$. Therefore $I'' \subseteq I'$, which proves that I' is open, as intended. □

Exercise 8.3. If X is a smooth vector field on a manifold M and $p \in M$ is a point where $X_p \neq 0$, then there exists a chart (U, ϕ) of M defined at p such $X|_U = \frac{\partial}{\partial \phi^0}$. ***Hint:*** It is easier to construct the inverse $\psi = \phi^{-1}$. Use a function of the form $\psi(x) = \Phi_X^{x^0}(f(x^1, \ldots, x^{n-1}))$, where $f : U \to M$ is a suitable function defined on an open set $U \subseteq \mathbb{R}^{n-1}$.

Solution. Let (V, η) be a chart centered at p, i.e. such that $\eta(p) = 0$. Denote X^i the components of X with respect to the chart η. Since $X|_p \neq 0$, we may assume w.l.o.g that $X^0 \neq 0$ at p, which means that the vectors $X|_p$, $\frac{\partial}{\partial \eta^1}\Big|_p, \ldots, \frac{\partial}{\partial \eta^{n-1}}\Big|_p$ are linearly independent.

Consider the map $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n : (x^1, \ldots, x^{n-1}) \mapsto (0, x^1, \ldots, x^{n-1})$, and let $W = \iota^{-1}(\eta(V))$, so that we can define the map $f = \eta^{-1} \circ \iota : W \to V$.

Define the map $\psi(x^0, x^1, \ldots, x^{n-1}) = \Phi_X^{x^0}(f(x^1, \ldots, x^{n-1}))$ at all points where the right hand side is defined. The domain of ψ is an open set which includes the slice $\{0_{\mathbb{R}}\} \times W$. The partial derivative of ψ with respect to x^0 is $\frac{\partial \psi(x^0, \ldots, x^{n-1})}{\partial x^0} = X$. Its other partial derivatives at the point $x = 0$ are

$$\left.\frac{\partial \psi(x^0, \ldots, x^{n-1})}{\partial x^i}\right|_{x=0} = \left.\frac{\partial f(x^1, \ldots, x^{n-1})}{\partial x^i}\right|_{x=0} = \left.\frac{\partial}{\partial \eta^i}\right|_p$$

for $i \neq 0$. Since the vectors X_p, $\frac{\partial}{\partial \eta^1}\Big|_p, \ldots, \frac{\partial}{\partial \eta^{n-1}}\Big|_p$ are linearly independent, we conclude that $\mathrm{T}_0\psi$ is an isomorphism. Thus there is a neighborhood Z of 0 in $\mathbb{R}^n$ such that the map $\psi|_Z : Z \to \psi(Z) \subseteq M$ is a diffeomorphism. Hence $\psi|_Z$ is a local parametrization of M. □

Exercise 8.4. (**To hand in**) Compute the flows of the following vector fields.

(a) On the plane $\mathbb{R}^2$, the "angular" vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.
(b) A constant vector field X on the torus $\mathbb{T}^n$.

Exercise 8.5. Let X be a $\mathcal{C}^\infty$ tangent vector field on a manifold M, with $k \geq 1$.

(a) For a point $p \in M$ and numbers $s, t \in \mathbb{R}$, show that the equation $\Phi_X^{(s+t)}(p) = \Phi_X^t(\Phi_X^s(p))$ holds if the right-hand side is defined.

Solution. Since we are only considering one vector field X, we may omit the subindex X and thus write $\Phi := \Phi_X$, $I_p := I_{X,p}$ and $\gamma_p := \gamma_{X,p}$.

The right-hand side is defined if and only if $s \in I_p$ (so that $q := \Phi^s(p)$ is defined) and $t \in I_q$ (so that $\Phi^t(q) = \Phi^t_X(\Phi^s(p))$ is defined). We assume this is the case.

The function $\tau \mapsto \gamma_p(\tau+s)$, defined for $\tau \in I_p - s$, is the curve γ_q, because it is a maximal integral curve of X that visits at time $\tau = 0$ the point $\gamma_p(s) = q$. In particular $I_q = I_p - s$. Thus since $t \in I_q$, it follows that $t + s \in I_p$, and that

$$\Phi^t(\Phi^s(p)) = \Phi^t(q) = \gamma_q(t) = \gamma_p(t+s) = \Phi^{t+s}(p).$$

$\square$

(b) We say that X is **complete** if its flow Φ_X is defined over $M \times \mathbb{R}$. Show that a compactly supported vector field is complete. In particular, on a compact manifold, every vector field is complete.

Solution. We will first show that the flow Φ_X is defined on a set $M \times (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Once this is established, we see that $\Phi^t_X(p)$ is defined for any $(p,t) \in M \times \mathbb{R}$ by decomposing $t = \sum_i t_i$ with $|t_i| < \varepsilon$ and applying the last formula $\Phi^t_X(p) = \Phi^{t_0}_X(\Phi^{t_1}_X \dots (p))$. This shows that X is complete.

Suppose X vanishes outside a compact set $K \subseteq M$. The domain of the flow Φ_X is a set $\mathrm{Dom}(\Phi_X) \subseteq M \times \mathbb{R}$ that contains the set $K \times \{0_{\mathbb{R}}\}$. In addition, $\mathrm{Dom}(\Phi_X)$ is open (this is part of the theorem of differentiability of the flow Φ_X). Since K is compact, by the *tube lemma* the open set $\mathrm{Dom}(\Phi_X)$ also contains a "tube neighborhood" $K \times (-\varepsilon, \varepsilon)$ of the set $K \times \{0_{\mathbb{R}}\}$, for some number $\varepsilon > 0$. But the domain $\mathrm{Dom}(\Phi_X)$ also contains the set $(M \setminus K) \times \mathbb{R}$, because for points $p \in M \setminus K$, since the vector field X vanishes at p, the maximal solution is the constant curve $\gamma_{X,p}(t) = p$, which is defined for all $t \in \mathbb{R}$. This shows that Φ_X is defined on $M \times (-\varepsilon, \varepsilon)$, and therefore Φ_X is complete, as explained above. $\square$

(c) If X is complete, show that the map Φ^t_X is a diffeomorphism $M \to M$.

Solution. Φ^t_X is a diffeomorphism with inverse Φ^{-t}_X since $\Phi^t_X \circ \Phi^{-t}_X = \Phi^{t-t} = \Phi^0 = \mathrm{id}_M$ and similarly $\Phi^{-t}_X \circ \Phi^t_X = \mathrm{id}_M$. $\square$

Exercise 8.6. If X is a complete $\mathcal{C}^\infty$ vector field with $(k \geq 1)$ and $h \in \mathcal{C}^\infty(M, \mathbb{R})$.

(a) Show that the function $X(h) : M \to \mathbb{R}$ that sends $p \mapsto X_p(h)$ is $\mathcal{C}^\infty$.

Solution. Take a chart (U, φ) and write $X|_U = \sum_i X^i \frac{\partial}{\partial \varphi^i}$. Then $X|_p(h) = \sum_i X^i|_p \frac{\partial}{\partial \varphi^i}|_p h$. Thus to see that the function $X(h)$ is $\mathcal{C}^\infty$, it suffices to check that the functions X^i and $\frac{\partial}{\partial \varphi^i} h$ are $\mathcal{C}^\infty$. And indeed: the fact that X is $\mathcal{C}^\infty$ means that the functions X^i are $\mathcal{C}^\infty$, and the fact that h is $\mathcal{C}^{k+1}$ implies that its first-order derivatives $\frac{\partial h}{\partial \varphi^i}$ are $\mathcal{C}^\infty$. $\square$

(b) Show that $X(h) = \frac{\partial}{\partial t}\big|_{t=0} h_t$, where $h_t := (\Phi^t_X)^*(h) = h \circ \Phi^t_X$.
Also show that $X(h_t) = (\Phi^t_X)^*(X(h))$.

Solution. Writing $\Phi := \Phi_X$, we have

$$\frac{\partial}{\partial t}\bigg|_{t=0} h_t(p) = \frac{\partial}{\partial t}\bigg|_{t=0} h\left(\Phi^t(p)\right) = \mathrm{T}_{\Phi^0(p)} h \left(\frac{\partial}{\partial t}\bigg|_{t=0} \Phi^t(p) \right) = D_p h(X_p) = X_p(h).$$

$\square$