Final Exam

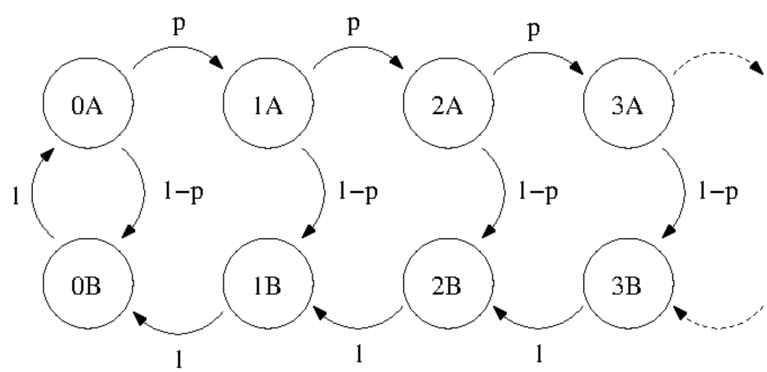
Surname: .... First name: .... Section: ....

Exercise 1. (22+2 points)

Useful reminders for this exercise: For any $0 < x < 1$ and $k \geq 1$, we have:

$$\sum_{n\geq 0} x^n = \frac{1}{1-x} \quad // \quad \sum_{n\geq 1} n x^{n-1} = \frac{\partial}{\partial x}\left(\sum_{n\geq 1} x^n\right) = \dots \quad // \quad \sum_{j=1}^{k} x^j = \frac{x^{k+1}-x}{x-1}$$

Let us consider the Markov chain $(X_n,\, n \geq 0)$ with state space $S = \{iA, iB,\, i \in \mathbb{N}\}$ and the following transition graph:



where $0 < p < 1$ is a fixed parameter.

a) For every $n \geq 1$, compute the value of

$$f^{(n)}_{0A,0A} = \mathbb{P}(X_n = 0A, X_{n-1} \neq 0A, \dots, X_1 \neq 0A \,|\, X_0 = 0A)$$

b) For what values of $0 < p < 1$ is state $0A$ recurrent? Justify your answer.

Let now $T_{0A} = \inf\{n \geq 1 \,:\, X_n = 0A\}$ be the first return time to state $0A$.

c) Compute $\mathbb{E}(T_{0A}\,|X_0 = 0A)$.

d) For what values of $0 < p < 1$ is state $0A$ positive-recurrent? Justify your answer.

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e) Without doing any computation, explain why does the chain $(X_n,\, n \geq 0)$ admit a unique stationary distribution π for every value of $0 < p < 1$.

f) Show by induction on i that $\pi_{iA} = \pi_{iB}$ for every $i \in \mathbb{N}$.

g) Use f) to compute the stationary distribution π.

h) Are the detailed balance equations satisfied? Justify your answer.

BONUS For every $n \geq 1$, compute the value of

$$p_{0A,0A}^{(n)} = \mathbb{P}(X_n = 0A \,|\, X_0 = 0A)$$

Exercise 2. (20+2 points)

Let $0 < p \leq \frac{1}{2}$ and $0 < q \leq 1$ be two fixed parameters and consider the Markov chain $(X_n,\, n \geq 0)$ with state space $\mathcal{S} = \{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 1-2p & p & p \\ q & 1-q & 0 \\ q & 0 & 1-q \end{pmatrix}$$

a) For any given values of p, q, compute the stationary distribution π of the chain X.

b) For any given values of p, q, compute the eigenvalues of P.

c) Deduce the corresponding spectral gap γ of the chain X, as well as a tight upper bound on

$$\|P_0^n - \pi\|_{\text{TV}}$$

for large values of n.

Let us now consider another Markov chain $(Y_n,\, n \geq 0)$ with same state space $\mathcal{S}$ and transition matrix

$$Q = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

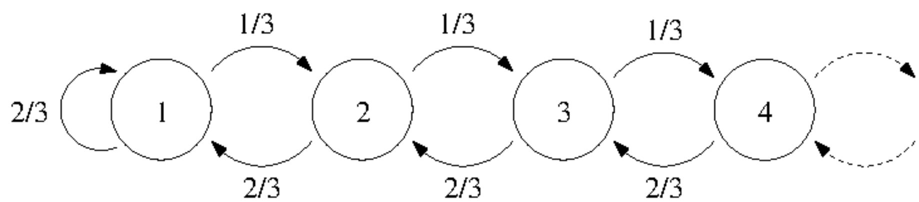
d) For what values of p, q do the chains X and Y share the same stationary distribution?

e) Among the values of p, q found in part d), which correspond to the largest spectral γ for the chain X?

BONUS Do the spectral gaps of X and Y match in this last case?

Exercise 3. (18 points)

Let us consider the Markov chain with state space $\mathcal{S} = \mathbb{N}^* = \{1, 2, 3, \ldots\}$, with transition graph



and with corresponding transition matrix Ψ.

a) Let $\pi = (\pi_1, \pi_2, \pi_3, \ldots)$ be a distribution on $\mathcal{S}$ such that $\pi_i > \pi_{i+1}$ for all $i \geq 1$. Starting from the base chain with transition matrix Ψ, design a new Markov chain chain with transition matrix P whose stationary distribution is π. Compute the matrix P explicitly.

b) What do we know about the chain with transition matrix P and the stationary distribution π? List all the properties you can think of.

c) Compute $\lim_{i\to\infty} p_{i,i+1}$ in the 3 following cases:

c1) $\pi_i = \frac{1}{Z}\,\frac{1}{i^q}$, $i \geq 1$. Here, $q > 1$ is a fixed parameter and $Z = \sum_{i\geq 1} \frac{1}{i^q}$.

c2) $\pi_i = \frac{1}{Z}\exp(-i)$, $i \geq 1$, with $Z = \sum_{i\geq 1} \exp(-i)$.

c3) $\pi_i = \frac{1}{Z}\exp(-i^2)$, $i \geq 1$, with $Z = \sum_{i\geq 1} \exp(-i^2)$.

d) For which of the above 3 example(s) does the Metropolis algorithm always accept a move from i to $i-1$, $\forall i \geq 2$?