Final Exam

Surname: .... First name: .... Section: ....

Quiz. True or false? (10 points; correct answer = 2pts, wrong answer = -1pt, no answer = 0pt)

a) A stationary distribution satisfying detailed balance is always also a limiting distribution.
☐ True ☐ False

b) If X, Y are two independent random variables with different distributions, then $\mathbb{P}(X \neq Y) > 0$.
☐ True ☐ False

c) If a Markov chain is finite, ergodic and reversible, then all the eigenvalues of its transition matrix are strictly less than 1 in absolute value.
☐ True ☐ False

d) The spectral gap of a finite, ergodic and reversible Markov chain characterizes always completely and precisely its convergence towards equilibirum.
☐ True ☐ False

e) The stationary distribution of a Metropolis chain necessarily satisfies detailed balance.
☐ True ☐ False

Exercise 1. (30 points)

Let $(X_n,\, n \geq 1)$ be a Markov chain with state space $\mathcal{S} = \{0, 1\}$, initial distribution $\pi^{(0)}$ and transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \quad \text{where} \quad 0 < p, q < 1$$

Let $(Y_n, n \geq 1)$ be another Markov chain with same state space $\mathcal{S}$ and same transition matrix P, but whose initial distribution π is also the stationary distribution of the chain.

a) Compute π.

We consider now the coupled chain $Z = (X, Y)$ with state space $\mathcal{S} \times \mathcal{S}$ such that X, Y evolve independently according to P as long as $X_n \neq Y_n$, and then evolve together, still according to P, as soon as $X_n = Y_n$.

b) Write down the transition matrix P_Z of the chain Z.

c) Which states of Z are transient / recurrent?

d) Does the chain Z admit a unique limiting and stationary distribution π_Z? If yes, compute it; if no, explain why.

e) Express $\mathbb{P}(X_{n+1} \neq Y_{n+1})$ as a function of $\mathbb{P}(X_n \neq Y_n)$.

f) From e), deduce an upper bound on $\max_{i \in \mathcal{S}} \|P_i^n - \pi\|_{\mathrm{TV}}$.

g) When $p = q$, what value of $0 < p < 1$ leads to the fastest convergence?

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Exercise 2. (16 points)

Let $M \geq 1$ be an integer and $N = 4M + 1$. We consider the Markov chain with state space $\mathcal{S} = \{0, 1, \dots, N-1\}$ and transition matrix given by

$$P = \text{circ}(0, 0, 1/2, 0, \dots, 0, 1/2, 0)$$

Note: In the case $N = 5$, $P = \text{circ}(0, 0, 1/2, 1/2, 0)$.

a) Compute the spectral gap of the chain.

Reminder: If A is an $N \times N$ circulant matrix $A = \text{circ}(c_0, c_1, \dots, c_{N-1})$, then its eigenvalues are given by

$$\lambda_k = \sum_{j=0}^{N-1} c_j \exp\left(\frac{2\pi i j k}{N}\right) \qquad k = 0, \dots, N-1$$

Consider now the modified chain with added self-loops, all of the same weight $0 < \alpha < 1$, and transition probabilities decreased accordingly by a multiplicative factor $1 - \alpha$.

b) For a fixed value of N, compute the value of α leading to the largest possible value of the spectral gap for the modified chain.

c) Compute the numerical value of α in the case $N = 5$.

Hints: Use the formula $\cos(2x) = 2\cos^2(x) - 1$ and the fact that $\cos(\pi/5) = \frac{1+\sqrt{5}}{4}$.

Exercise 3. (16 points)

Let first $i_0 \in \mathbb{Z}$ and $\beta > 0$. Let us then consider the Metropolis chain with state space $\mathcal{S} = \mathbb{Z}$ obtained from the base chain with the following transition probabilities:

$$\psi_{ij} = \begin{cases} 1/2 & \text{if } |j - i| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and with the following acceptance probabilities:

$$a_{ij} = \begin{cases} e^{-\beta} & \text{if } (j = i + 1 \text{ and } i \geq i_0) \text{ or } (j = i - 1 \text{ and } i \leq i_0) \\ 1 & \text{if } (j = i + 1 \text{ and } i < i_0) \text{ or } (j = i - 1 \text{ and } i > i_0) \end{cases}$$

a) Compute the limiting distribution π of the Metropolis chain.

b) Assuming first that the Metropolis chain is run for a sufficiently long time, what is the probability that state i_0 is sampled with this method?

c) For what values of $\beta > 0$ is this probability greater than or equal to 1/2 ?

d) Assume now that $i_0 > 0$, that the Metropolis chain starts in position $i = 0$ with probability 1, and that it is run only for i_0 steps. What is the probability that state i_0 is sampled in this case?