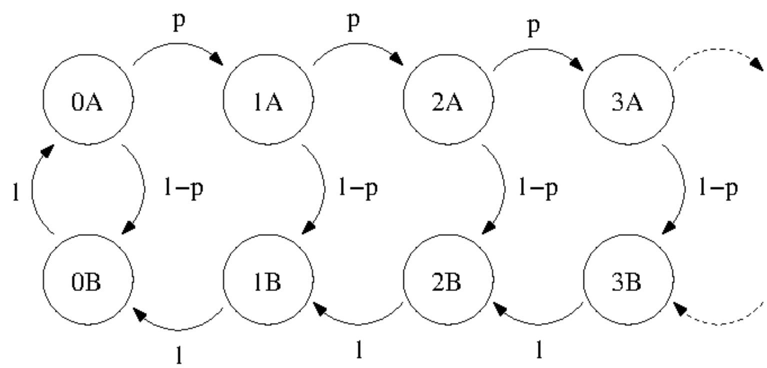
Final Exam

Exercise 1. (22+2 points)

Useful reminders for this exercise: For any $0 < x < 1$ and $k \geq 1$, we have:

$$\sum_{n\geq 0} x^n = \frac{1}{1-x} \quad // \quad \sum_{n\geq 1} n x^{n-1} = \frac{\partial}{\partial x}\left(\sum_{n\geq 1} x^n\right) = \dots \quad // \quad \sum_{j=1}^{k} x^j = \frac{x^{k+1}-x}{x-1}$$

Let us consider the Markov chain $(X_n,\, n \geq 0)$ with state space $S = \{iA, iB,\, i \in \mathbb{N}\}$ and the following transition graph:



where $0 < p < 1$ is a fixed parameter.

a) For every $n \geq 1$, compute the value of

$$f^{(n)}_{0A,0A} = \mathbb{P}(X_n = 0A, X_{n-1} \neq 0A, \dots, X_1 \neq 0A \,|\, X_0 = 0A)$$

Answer: Any path starting from the state OA and returning to it has even length. Therefore $f^{(n)}_{0A,0A} = 0$ for n odd. If n is even then the only path starting from $0A$ and returning to it for the first time after n steps is

$$0A \to 1A \to \dots \to (n/2-1)A \to (n/2-1)B \to 0B \to 0A\,.$$

It comes $f^{(n)}_{0A,0A} = (1-p)p^{\frac{n}{2}-1}$ for $n \geq 1$ even.

b) For what values of $0 < p < 1$ is state $0A$ recurrent? Justify your answer.

Answer: For $0 < p < 1$

$$\sum_{n=1}^{+\infty} f^{(n)}_{0A,0A} = \sum_{m=1}^{+\infty} (1-p)p^{\frac{2m}{2}-1} = (1-p)\sum_{m=1}^{+\infty} p^{m-1} = (1-p)\cdot\frac{1}{1-p} = 1\,.$$

The state $0A$ is therefore recurrent for all $p \in (0,1)$.

Let now $T_{0A} = \inf\{n \geq 1 \,:\, X_n = 0A\}$ be the first return time to state $0A$.

c) Compute $\mathbb{E}(T_{0A}\,|X_0 = 0A)$.

Answer: Remember that $\mathbb{P}(T_{0A} = n|X_0 = 0A) = f^{(n)}_{0A,0A}$. Then the expected return time is computed by making use of the second formula given in introduction:

$$\mathbb{E}(T_{0A}\,|X_0 = 0A) = \sum_{n=1}^{+\infty} n f^{(n)}_{0A,0A} = \sum_{m=1}^{+\infty} 2m(1-p)p^{m-1} = 2(1-p)\frac{\partial}{\partial x}\left(\sum_{m\geq 1} x^m\right)\bigg|_{x=p} = \frac{2}{1-p}\,.$$

The last equality follows from $\sum_{m\geq 1} x^m = \frac{1}{1-x}$ and $\frac{\partial}{\partial x}\left(\frac{1}{1-x}\right) = (1-x)^{-2}$.

d) For what values of $0 < p < 1$ is state $0A$ positive-recurrent? Justify your answer.

Answer: The expected return time is finite, and therefore the state $0A$ is positive-recurrent, for all $p \in (0,1)$.

e) Without doing any computation, explain why does the chain $(X_n,\, n \geq 0)$ admit a unique stationary distribution π for every value of $0 < p < 1$.

Answer: The chain is irreducible and positive-recurrent (we proved $0A$ is positive-recurrent and the chain has a unique equivalence class). By a theorem seen in class the existence and uniqueness of a stationary distribution π follows.

f) Show by induction on i that $\pi_{iA} = \pi_{iB}$ for every $i \in \mathbb{N}$.

Answer: The equation $\pi = \pi P$ reads

$$\pi_{0A} = \pi_{OB} \quad \text{and} \quad \forall i \geq 1 : \pi_{iA} = p\pi_{(i-1)A}\,,\ \pi_{(i-1)B} = (1-p)\pi_{(i-1)A} + \pi_{iB}\,.$$

Hence $\pi_{iB} - \pi_{iA} = \pi_{(i-1)B} - (1-p)\pi_{(i-1)A} - p\pi_{(i-1)A} = \pi_{(i-1)B} - \pi_{(i-1)A}$ for all $i \geq 1$. The result follows by induction, as $\pi_{0A} = \pi_{OB}$.

g) Use f) to compute the stationary distribution π.

Answer: We have seen in class that $\pi_{0A} = \frac{1}{\mathbb{E}(T_{0A}\,|X_0=0A)} = \frac{1-p}{2}$. Besides, for all $i \geq 1$, $\pi_{iA} = \pi_{iB}$ and $\pi_{iA} = p\pi_{(i-1)A} = p^i\pi_{0A}$.The stationary distribution follows:

$$\forall i \in \mathbb{N}:\ \pi_{iA} = \pi_{iB} = \frac{p^i\,(1-p)}{2}\,.$$

h) Are the detailed balance equations satisfied?

Answer: The detailed balance equations are not satisfied, because there exist states i, j with $p_{ij} > 0$ and $p_{ji} = 0$.

BONUS For every $n \geq 1$, compute the value of

$$p^{(n)}_{0A,0A} = \mathbb{P}(X_n = 0A\,|\,X_0 = 0A)$$

Answer: One sees easily that $p^{(n)}_{0A,0A} = 0$ for n odd and that $p^{(2)}_{0A,0A} = 1-p$. Likewise, direct computations show that $p^{(4)}_{0A,0A} = p^{(6)}_{0A,0A} = 1-p, \ldots$, so this suggests trying to prove by induction that $p^{(n)}_{0A,0A} = 1-p$ for all even n's. Assume indeed $p^{(2n)}_{0A,0A} = 1-p$ for all $1 \leq k \leq n$ (remembering that $p^{(0)}_{0A,0A} = 1$ by convention). From the course, we know that

$$\begin{aligned} p^{(2n+2)}_{0A,0A} &= \sum_{k=1}^{n+1} f^{(2k)}_{0A,0A}\, p^{(2n+2-2k)}_{0A,0A} = \sum_{k=1}^{n} p^{k-1}\,(1-p)\,(1-p) + p^n\,(1-p)\,1 \\ &= \frac{p^n - p}{p-1}\,(1-p)^2 + p^n\,(1-p) = 1-p \end{aligned}$$

which proves the claim.

Exercise 2. (20+2 points) Let $0 < p \leq \frac{1}{2}$ and $0 < q \leq 1$ be two fixed parameters and consider the Markov chain $(X_n,\, n \geq 0)$ with state space $\mathcal{S} = \{0,1,2\}$ and transition matrix

$$P = \begin{pmatrix} 1-2p & p & p \\ q & 1-q & 0 \\ q & 0 & 1-q \end{pmatrix}$$

a) For any given values of p, q, compute the stationary distribution π of the chain X.

Answer: The system of equations for the stationary distribution π reads

$$\pi = \pi P \Leftrightarrow \begin{cases} \pi_0 = (1-2p)\pi_0 + q\pi_1 + q\pi_2 \\ \pi_1 = p\pi_0 + (1-q)\pi_1 \\ \pi_2 = p\pi_0 + (1-q)\pi_2 \end{cases} \Leftrightarrow \pi_1 = \pi_2 = \frac{p}{q}\pi_0\,.$$

This last equation, combined with $\pi_0 + \pi_1 + \pi_1 = 1$, gives $\pi_0 = \frac{q}{q+2p}$, $\pi_1 = \pi_2 = \frac{p}{q+2p}$.

b) For any given values of p, q, compute the eigenvalues of P.

Answer: One of the eigenvalue is of course $\lambda_0 = 1$. Besides the eigenvalues satisfy

$$\begin{aligned} \mathrm{Tr}(P) &= \lambda_0 + \lambda_1 + \lambda_2 = 3 - 2p - 2q\,, \\ \det(P) &= \lambda_0\lambda_1\lambda_2 = (1-2p)(1-q)^2 - 2pq(1-q) = (1-q)(1-2p-q)\,. \end{aligned}$$

λ_1,λ_2 are the roots of the polynomial $X^2 - 2(1-p-q)X + (1-q)(1-2p-q)$ whose discriminant is $\Delta = 4(1-p-q)^2 - 4(1-q)(1-2p-q) = 4p^2$. We find

$$\lambda_1 = 1-q\,,\ \lambda_2 = 1-2p-q\,.$$

c) Deduce the corresponding spectral gap γ of the chain X, as well as a tight upper bound on

$$\|P_0^n - \pi\|_{\mathrm{TV}}$$

for large values of n.

Answer: The spectral gap is

$$\gamma = \max\{\lambda_1, -\lambda_2\} = \begin{cases} q & \text{if } p+q \leq 1\,; \\ 2(1-p)-q & \text{if } p+q > 1\,. \end{cases}$$

For large n the upperbound $\|P_0^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2\sqrt{\pi_0}} e^{-n\gamma}$ is tight.

Let us now consider another Markov chain $(Y_n,\, n \geq 0)$ with same state space $\mathcal{S}$ and transition matrix

$$Q = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

d) For what values of p, q do the chains X and Y share the same stationary distribution?

Answer: Q is doubly stochastic, so its stationary distribution is the uniform distribution on $\mathcal{S}$, i.e., $(1/3, 1/3, 1/3)$. Clearly, $\pi_0 = \pi_1 = \pi_2$ if and only if $0 < p = q \leq \frac{1}{2}$.

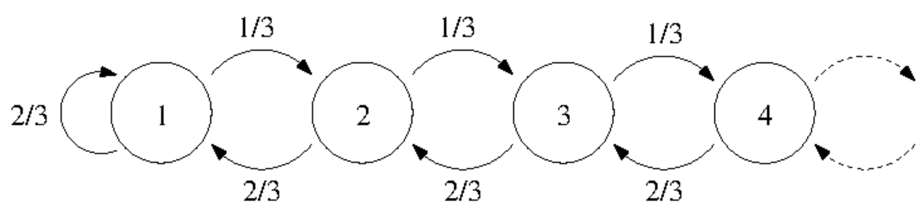
e) Among the values of p, q found in part d), which correspond to the largest spectral γ for the chain X?

Answer: If $0 < p = q \leq \frac{1}{2}$ then $p + q = 2p \leq 1$ and the spectral gap is $\gamma = q$. It is the largest when $p = q = \frac{1}{2}$.

BONUS Do the spectral gaps of X and Y match in this last case?

Answer: The eigenvalues of Q are 1 with geometric multiplicity 1 and $-\frac{1}{2}$ with geometric multiplicity 2 (the eigenvectors are easily seen to be $(1, -1, 0)^T$ and $(0, 1, -1)^T$). Then the spectral gap of Y is $\frac{1}{2}$, which matches with the spectral gap of X in this last case.

Exercise 3. (18 points) Let us consider the Markov chain with state space $\mathcal{S} = \mathbb{N}^* = \{1, 2, 3, \ldots\}$, with transition graph



and with corresponding transition matrix Ψ.

a) Let $\pi = (\pi_1, \pi_2, \pi_3, \ldots)$ be a distribution on $\mathcal{S}$ such that $\pi_i > \pi_{i+1}$ for all $i \geq 1$. Starting from the base chain with transition matrix Ψ, design a new Markov chain chain with transition matrix P whose stationary distribution is π. Compute the matrix P explicitly.

Answer: We use the Metropolis-Hasting algorithm to build a new chain which satisfies the detailed balance equations for π (making π a stationary disitribution for this chain). The acceptance probabilities will be $a_{ij} = \min\left\{1, \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij}}\right\}$ whenever $\psi_{ij} > 0$ (note that $\psi_{ij} \neq 0 \Leftrightarrow \psi_{ji} \neq 0$). The transition probabilities of the matrix P read

$$p_{ij} = \begin{cases} \frac{1}{3}\min\left\{1, \frac{2\pi_{i+1}}{\pi_i}\right\} & \text{for } j = i+1, i \geq 1\,; \\ \frac{2}{3}\min\left\{1, \frac{\pi_{i-1}}{2\pi_i}\right\} & \text{for } j = i-1, i \geq 2\,; \\ 1 - \frac{1}{3}\min\left\{1, \frac{2\pi_{i+1}}{\pi_i}\right\} - \frac{2}{3}\min\left\{1, \frac{\pi_{i-1}}{2\pi_i}\right\} & \text{for } j = i, i \geq 2\,; \\ 1 - \frac{1}{3}\min\left\{1, \frac{2\pi_2}{\pi_1}\right\} & \text{for } j = i = 1\,. \end{cases}$$

b) What do we know about the chain with transition matrix P and the stationary distribution π? List all the properties you can think of.

Answer: The base chain is irreducible, aperiodic, and so is the new chain. Besides, the new chain is built to satisfy the detailed balance equations so that π is its stationary distribution. Hence the new chain is positive-recurrent (it is irreducible and has a stationary distribution). We have all the properties to conclude that the chain with transition matrix P is ergodic (so that the stationary distribution π is also a limiting distribution).

c) Compute $\lim_{i\to\infty} p_{i,i+1}$ in the 3 following cases:

c1) $\pi_i = \frac{1}{Z}\frac{1}{i^q}$, $i \geq 1$. Here, $q > 1$ is a fixed parameter and $Z = \sum_{i\geq 1}\frac{1}{i^q}$.

c2) $\pi_i = \frac{1}{Z}\exp(-i)$, $i \geq 1$, with $Z = \sum_{i\geq 1}\exp(-i)$.

c3) $\pi_i = \frac{1}{Z}\exp(-i^2)$, $i \geq 1$, with $Z = \sum_{i\geq 1}\exp(-i^2)$.

Answer: From question **a)**, $p_{i,i+1} = \frac{1}{3}\min\left\{1, \frac{2\pi_{i+1}}{\pi_i}\right\}$. It comes

$$\begin{aligned} \textbf{c1)}\ p_{i,i+1} &= \frac{1}{3}\min\left\{1, \frac{2i^q}{(i+1)^q}\right\} = \frac{1}{3}\min\left\{1, \frac{2}{(1+i^{-1})^q}\right\} \to \frac{1}{3}\,; \\ \textbf{c2)}\ p_{i,i+1} &= \frac{1}{3}\min\left\{1, \frac{2e^{-i-1}}{e^{-i}}\right\} = \frac{1}{3}\min\left\{1, 2e^{-1}\right\} = \frac{2}{3e} \to \frac{2}{3e}\,; \\ \textbf{c3)}\ p_{i,i+1} &= \frac{1}{3}\min\left\{1, \frac{2e^{-(i+1)^2}}{e^{-i^2}}\right\} = \frac{1}{3}\min\left\{1, 2e^{-1-2i}\right\} \to 0\,. \end{aligned}$$

d) For which of the above 3 example(s) does the Metropolis algorithm always accept a move from i to $i-1$, $\forall i \geq 2$?

Answer: From question **a)**, $a_{i,i-1} = \min\left\{1, \frac{\pi_{i-1}}{2\pi_i}\right\}$ for $i \geq 2$. It comes

$$\begin{aligned} \textbf{c1)}\ a_{i,i-1} &= \min\left\{1, \frac{i^q}{2(i-1)^q}\right\} = \min\left\{1, \frac{1}{2(1-i^{-1})^q}\right\} < 1 \text{ for large values of } i\,; \\ \textbf{c2)}\ a_{i,i-1} &= \min\left\{1, \frac{e^{-i+1}}{2e^{-i}}\right\} = \min\left\{1, e/2\right\} = 1\,; \\ \textbf{c3)}\ a_{i,i-1} &= \min\left\{1, \frac{e^{-(i-1)^2}}{2e^{-i^2}}\right\} = \min\left\{1, e^{-1+2i}/2\right\} = 1\,. \end{aligned}$$

So only in the last two cases does the Metropolis algorithm always accept a move from i to $i-1$, $\forall i \geq 2$.