Final Exam Solutions

Quiz. True or false? (10 points)

a) False. It has to be aperiodic too.

b) True.

c) False. We always have $\lambda_0 = 1$.

d) False. It cannot explain e.g. the cut-off phenomenon.

e) True.

Exercise 1. (30 points)

a) With some basic calculation, we obtain

$$\pi_0 = \frac{q}{p+q} \quad \text{and} \quad \pi_1 = \frac{p}{p+q}$$

b) The transition matrix over the state space $\mathcal{S} \times \mathcal{S} = \{(0,0),(0,1),(1,0),(1,1)\}$ is given by

$$P_Z = \begin{pmatrix} 1-p & 0 & 0 & p \\ (1-p)q & (1-p)(1-q) & pq & p(1-q) \\ (1-p)q & pq & (1-p)(1-q) & p(1-q) \\ q & 0 & 0 & 1-q \end{pmatrix}$$

c) States $(0,1)$ and $(1,0)$ are transient, and states $(0,0)$ and $(1,1)$ are recurrent.

d) Z admits a unique stationary distribution, which is zero over the transient states and is equal to π for the recurrent states (similar to what we saw in the homeworks). Formally, the stationary distribution of Z is

$$\pi_Z = \left(\frac{q}{p+q}, 0, 0, \frac{p}{p+q}\right)$$

e) Let us define the event $A_n = \{X_n \neq Y_n\}$. Then we have

$$\mathbb{P}(A_{n+1}) = \mathbb{P}(A_{n+1}|A_n)\,\mathbb{P}(A_n) + \mathbb{P}(A_{n+1}|A_n^c)\,\mathbb{P}(A_n^c) = \mathbb{P}(A_{n+1}|A_n)\,\mathbb{P}(A_n)$$

where we used the fact that, due the definition of the coupling, we have $\mathbb{P}(A_{n+1}|A_n^c) = 0$. Then, for the term $\mathbb{P}(A_{n+1}|A_n)$, we have

$$\begin{aligned}\mathbb{P}(A_{n+1}|A_n) &= \mathbb{P}(A_{n+1}|Z_n = (0,1))\,\mathbb{P}(Z_n = (0,1)|A_n) + \mathbb{P}(A_{n+1}|Z_n = (1,0))\,\mathbb{P}(Z_n = (1,0)|A_n) \\ &= \Big(pq + (1-p)(1-q)\Big)\Big(\mathbb{P}(Z_n = (0,1)|A_n) + \mathbb{P}(Z_n = (1,0)|A_n)\Big) = pq + (1-p)(1-q)\end{aligned}$$

where we used P_Z from part b). Overall,

$$\mathbb{P}(X_{n+1} \neq Y_{n+1}) = \Big(pq + (1-p)(1-q)\Big)\,\mathbb{P}(X_n \neq Y_n)$$

f) Based on the material of the course, if X_0 has an initial distribution $\pi^{(0)}$ and Y_0 has the stationary distribution π as its initial distribution, we obtain

$$\|\pi^{(n)} - \pi\|_{\text{TV}} \leq \mathbb{P}(X_n \neq Y_n) = \Big(pq + (1-p)(1-q)\Big)\, \mathbb{P}(X_{n-1} \neq Y_{n-1})$$

which naturally leads to

$$\|\pi^{(n)} - \pi\|_{\text{TV}} \leq \Big(pq + (1-p)(1-q)\Big)^n\, \mathbb{P}(X_0 \neq Y_0)$$

Then, considering $\pi^{(0)}$ as a distribution concentrated on a single state, we obtain

$$\max_{i\in\mathcal{S}} \|P_i^n - \pi\|_{TV} \leq \Big(pq + (1-p)(1-q)\Big)^n \max\left\{\frac{q}{p+q}, \frac{p}{p+q}\right\} \leq \Big(pq + (1-p)(1-q)\Big)^n$$

g) Let us define the function $f(p) = p^2 + (1-p)^2$. Then, for the case that $p = q$, the upper bound found in the previous part can be written as

$$\max_{i\in\mathcal{S}} \|P_i^n - \pi\|_{\text{TV}} \leq f(p)^n$$

Hence, the fastest convergence corresponds to the minimum value of $f(p)$, which can be found by solving

$$f'(p) = 2p - 2(1-p) = 0$$

whose solution is $p^* = \frac{1}{2}$.

Exercise 2. (16 points)

a) The eigenvalues of the mentioned transition matrix can be computed as follows:

$$\lambda_k = \frac{1}{2}\exp\left(i\frac{4\pi k}{N}\right) + \frac{1}{2}\exp\left(-i\frac{4\pi k}{N}\right) = \cos\left(\frac{4\pi k}{N}\right)$$

By quickly checking the values $k = 1$, $k = M$, $k = M+1$, $k = 2M$, $k = 2M+1$, $k = 3M$, $k = 3M+1$, and $k = 4M$ (i.e., the ones for which λ_k can be closest to 1 or -1), we see that

$$\lambda_{\text{min}} = -\cos\left(\frac{\pi}{N}\right) \quad \text{and} \quad \lambda_{\text{max}} = \cos\left(\frac{2\pi}{N}\right)$$

The spectral gap is then $\gamma = 1 - \cos\left(\frac{\pi}{N}\right)$.

Note. For odd values of N, the graph of this chain is actually isomorphic to the one encountered in Homework 6, Exercise 1. So the results are actually all the same!

b) By adding self loops, we end up with

$$\lambda'_{\text{min}} = \alpha - (1-\alpha)\cos\left(\frac{\pi}{N}\right) \quad \text{and} \quad \lambda'_{\text{max}} = \alpha + (1-\alpha)\cos\left(\frac{2\pi}{N}\right)$$

The maximum spectral gap is corresponding to the case where $|\lambda'_{\text{min}}| = \lambda'_{\text{max}}$, which corresponds to

$$\alpha^* = \frac{\cos\left(\frac{\pi}{N}\right) - \cos\left(\frac{2\pi}{N}\right)}{2 + \cos\left(\frac{\pi}{N}\right) - \cos\left(\frac{2\pi}{N}\right)}$$

c) For $N = 5$, we need to compute $\cos\left(\frac{\pi}{5}\right)$ and $\cos\left(\frac{2\pi}{5}\right)$. Given the hint, we have

$$\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4} \quad \text{and} \quad \cos\left(\frac{2\pi}{5}\right) = 2\cos^2\left(\frac{\pi}{5}\right) - 1 = \frac{-1+\sqrt{5}}{4}$$

so $\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{2\pi}{5}\right) = \frac{1}{2}$, which leads to $\alpha^* = \frac{1/2}{2+1/2} = \frac{1}{5}$.

Exercise 3. (16 points)

a) According to the Metropolis Hasting algorithm, we have

$$a_{ij} = \min\left\{1, \frac{\pi_j}{\pi_i}\frac{\psi_{ji}}{\psi_{ij}}\right\}$$

Hence:

$$\begin{aligned} &\text{if } i \geq i_0 \quad \text{then } \pi_{i+1} = e^{-\beta}\,\pi_i \\ &\text{if } i \leq i_0 \quad \text{then } \pi_{i-1} = e^{-\beta}\,\pi_i \end{aligned}$$

Therefore, for any $n \geq 0$, we obtain

$$\pi_{i_0+n} = \pi_{i_0-n} = e^{-n\beta}\,\pi_{i_0}$$

In addition, by the fact that the probabilities should sum to one, we have

$$\sum_i \pi_i = \pi_{i_0} + \sum_{n\neq 0}\pi_{i_0+n} = \pi_{i_0} + 2\sum_{n>0}\pi_{i_0}e^{-n\beta} = \pi_{i_0}\left(1 + 2\,\frac{e^{-\beta}}{1-e^{-\beta}}\right) = 1$$

and

$$\pi_{i_0} = \frac{1-e^{-\beta}}{1+e^{-\beta}}$$

As a result, we obtain

$$\pi_i = \frac{1-e^{-\beta}}{1+e^{-\beta}}\,e^{-|i-i_0|\beta}$$

b) Since the chain is ergodic, we have

$$\lim_{n\to\infty}\mathbb{P}(X_n = i_0) = \pi_{i_0} = \frac{1-e^{-\beta}}{1+e^{-\beta}}$$

c) Using the results of part b), we obtain

$$\frac{1-e^{-\beta}}{1+e^{-\beta}} \geq \frac{1}{2}$$

which leads to $\beta \geq \ln(3)$.

d) To obtain $X_{i_0} = i_0 > 0$ given $X_0 = 0$, the chain should go to the right for all steps before i_0, which means that

$$\mathbb{P}(X_{i_0} = i_0 | X_0 = 0) = \prod_{i=1}^{i_0}\mathbb{P}(X_i = i | X_{i-1} = i-1) = \prod_{i=1}^{i_0}\frac{1}{2} = \frac{1}{2^{i_0}}$$