Final Exam

Surname: .... First name: .... Section: ....

Exercise 1. (12 points) For each statement below, tell whether it is true or false, and provide a justification if the answer is "true" / a counter-example if the answer is "false".

a) Any finite Markov chain admits at least one stationary distribution.

b) If P is the transition matrix of a Markov chain, then $\frac{P+P^T}{2}$ is also a transition matrix.

c) If P is the transition matrix of a 3-periodic Markov chain, then $P^3 = I$.

d) Let X, Y be two irreducible Markov chains evolving independently on a *finite* common state space $\mathcal{S}$, according to a common transition matrix P, but with two different initial distributions. Then $\mathbb{P}(\exists n \geq 1 \text{ such that } X_n = Y_n) = 1$.

Exercise 2. (20 points) Let us consider the Markov chain $(X_n,\, n \geq 0)$ with state space

$$\mathcal{S} = \{0\} \cup \{iA,\, i \in \mathbb{N}^*\} \cup \{iB,\, i \in \mathbb{N}^*\} \cup \{iC,\, i \in \mathbb{N}^*\} \quad (\text{where } \mathbb{N}^* = \{1, 2, 3, \ldots\})$$

and transition matrix P given by

$$p_{0,1A} = p_{0,1B} = p_{0,1C} = \frac{1}{3} \quad p_{1A,0} = p_{1B,0} = p_{1C,0} = \frac{2}{3}$$

and for $i \in \mathbb{N}^*$:

$$p_{iA,(i+1)A} = p_{iB,(i+1)B} = p_{iC,(i+1)C} = \frac{1}{3} \quad p_{(i+1)A,iA} = p_{(i+1)B,iB} = p_{(i+1)C,iC} = \frac{2}{3}$$

a) Is the chain X irreducible? aperiodic?

b) Compute the unique stationary distribution π of the chain X. Is the detailed balance equation satisfied?

c) Prove that the chain X is positive-recurrent.

d) Compute $\mathbb{E}(T_0 \,|\, X_0 = 0)$, where $T_0 = \inf\{n \geq 1 \,:\, X_n = 0\}$.

e) Is the distribution π also a limiting distribution for the chain X?

Consider now the chain $(Y_n,\, n \geq 0)$ with same state space $\mathcal{S}$ and transition matrix $Q = P^2$.

f) Compute all the transition probabilities in Q.

g) Is the chain Y irreducible? aperiodic?

h) Does it hold that the distribution π computed above is also a stationary distribution for the chain Y? If yes, is it the unique stationary distribution? Justify.

i) Compute $\lim_{n\to\infty} \mathbb{P}(Y_n = 0 \,|\, Y_0 = 0)$.

Exercise 3. (8 points) Let $(X_n,\, n \geq 0)$ be a Markov chain with state space $\mathcal{S} = \{-, 0, +\}$ and transition matrix satisfying

$$p_{+,-} = p_{-,+} = 0, \quad p_{+,0} = p_{-,0} = a \quad \text{and} \quad p_{0,+} = p_{0,-} = b/2$$

where $0 < a, b < 1$.

a) Complete the transition matrix P.

b) Compute its eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2$.

c) Deduce the value of the spectral gap γ.

d) Under which minimal condition on the parameters a, b does it hold that $\gamma = \lambda_0 - \lambda_1$?

Exercise 4. (20 points) We consider the Ising distribution on two vertices $\{1, 2\}$. Let $\underline{s} = (s_1, s_2) \in \{-1, +1\}^2$ and

$$\pi_\beta(\underline{s}) = \frac{\exp(\beta s_1 s_2)}{4\cosh(\beta)}$$

with $\beta \geq 0$. Recall that $\cosh(x) = \frac{1}{2}\left(\exp(x) + \exp(-x)\right)$.

Consider the Metropolis-Hastings Markov chain $(\underline{S}_n,\, n \geq 0)$ defined as follows (starting from the initial uniform distribution on $\{-1, +1\}^2$):

- at step n, select a vertex $v \in \{1, 2\}$ uniformly at random;
- propose the move $\underline{S}^{(n)} \to \underline{S}^{(n+1)}$, where $S_v^{(n)}$ is flipped with probability $0 < q \leq 1$;
- accept the move according to the usual Metropolis-Hastings rule.

a) Draw the transition graph of the Markov chain $(\underline{S}_n,\, n \geq 0)$.

Consider now the magnetization at time $n \geq 0$:

$$M_n = S_1^{(n)} + S_2^{(n)}$$

b) Explain why the process $(M_n,\, n \geq 0)$ is also a Markov chain.

c) Compute its state space $\mathcal{S}$ and its transition matrix P.

d) Show that the chain $(M_n,\, n \geq 0)$ is ergodic for $0 < q \leq 1$ and $0 \leq \beta < +\infty$. Compute its limiting and stationary distribution π.

e) Compute the spectral gap γ of the chain $(M_n,\, n \geq 0)$.

f) When $q = \frac{1}{2}$, deduce a tight upper bound on $\|P_0^n - \pi\|_{\text{TV}}$ for $n \geq 1$.

g) For what value of q (as a function of β) is the spectral gap γ maximized?