Final Exam: Solutions

Exercise 1. (12 points) For each statement below, tell whether it is true or false, and provide a justification if the answer is "true" / a counter-example if the answer is "false".

a) Any finite Markov chain admits at least one stationary distribution.

Answer: true. Any finite Markov chain can be decomposed into equivalence classes, and at least one of these classes is recurrent (not all can be transient, because the number of states is finite). This class being finite and irreducible, it admits a stationary distribution (and weight 0 can be attributed to all other states in the chain).

b) If P is the transition matrix of a Markov chain, then $\frac{P+P^T}{2}$ is also a transition matrix.

Answer: false. P^T might not be a transition matrix, same for $\frac{P+P^T}{2}$. Example: consider $P = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$; then $\frac{P+P^T}{2} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \end{pmatrix}$.

c) If P is the transition matrix of a 3-periodic Markov chain, then $P^3 = I$.

Answer: false. Consider the chain with $S = \{0, 1, 2, 3, 4\}$ and $p_{01} = p_{12} = p_{34} = p_{42} = 1$, and $p_{23} = p_{20} = \frac{1}{2}$. This chain is 3-periodic, but $P^3 \neq I$.

d) Let X, Y be two irreducible Markov chains evolving independently on a *finite* common state space $\mathcal{S}$, according to a common transition matrix P, but with two different initial distributions. Then $\mathbb{P}(\exists n \geq 1 \text{ such that } X_n = Y_n) = 1$.

Answer: false. Consider the chain with two states and $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and X starting in the first state, Y starting in the second. Then $\mathbb{P}(\exists n \geq 1 \text{ such that } X_n = Y_n) = 0$.

Exercise 2. (20 points) Let us consider the Markov chain $(X_n,\, n \geq 0)$ with state space

$$\mathcal{S} = \{0\} \cup \{iA,\, i \in \mathbb{N}^*\} \cup \{iB,\, i \in \mathbb{N}^*\} \cup \{iC,\, i \in \mathbb{N}^*\} \quad \text{(where } \mathbb{N}^* = \{1, 2, 3, \ldots\}\text{)}$$

and transition matrix P given by

$$p_{0,1A} = p_{0,1B} = p_{0,1C} = \frac{1}{3} \quad p_{1A,0} = p_{1B,0} = p_{1C,0} = \frac{2}{3}$$

and for $i \in \mathbb{N}^*$:

$$p_{iA,(i+1)A} = p_{iB,(i+1)B} = p_{iC,(i+1)C} = \frac{1}{3} \quad p_{(i+1)A,iA} = p_{(i+1)B,iB} = p_{(i+1)C,iC} = \frac{2}{3}$$

a) Is the chain X irreducible? aperiodic?

Answer: the chain is irreducible and periodic with period 2.

b) Compute the unique stationary distribution π of the chain X. Is the detailed balance equation satisfied?

Answer: $\pi_0 = \frac{1}{4}$ and $\pi_{iA} = \pi_{iB} = \pi_{iC} = \frac{1}{2^{i+2}}$ for $i \in \mathbb{N}^*$. Yes, detailed balance holds.

c) Prove that the chain X is positive-recurrent.

Answer: It is irreducible and admits a stationary distribution, so by the theorem seen in class, it is positive-recurrent.

d) Compute $\mathbb{E}(T_0 \,|\, X_0 = 0)$, where $T_0 = \inf\{n \geq 1 \,:\, X_n = 0\}$.

Answer: Again by the theorem seen in class, $\mathbb{E}(T_0 \,|\, X_0 = 0) = \frac{1}{\pi_0} = 4$.

e) Is the distribution π also a limiting distribution for the chain X?

Answer: No, because the chain X is periodic.

Consider now the chain $(Y_n,\, n \geq 0)$ with same state space $\mathcal{S}$ and transition matrix $Q = P^2$.

f) Compute all the transition probabilities in Q.

Answer: for $X = A, B$ or C and $i \geq 2$, we have

$$q_{iX,iX} = \frac{4}{9}, \quad q_{iX,(i+2)X} = \frac{1}{9}, \quad q_{iX,(i-2)X} = \frac{4}{9}$$

and for X, Y designing two different letters among A, B, C, we have

$$q_{1X,1X} = \frac{4}{9}, \quad q_{1X,3X} = \frac{1}{9}, \quad q_{1X,1Y} = \frac{2}{9}$$

and finally, we also have $q_{0,0} = \frac{2}{3}$ and $q_{0,2X} = \frac{1}{9}$.

g) Is the chain Y irreducible? aperiodic?

Answer: The chain Y has two equivalent classes (even and odd values if i), both of them are aperiodic (because of the self-loops).

h) Does it hold that the distribution π computed above is also a stationary distribution for the chain Y? If yes, is it the unique stationary distribution? Justify.

Answer: Yes: $\pi P^2 = (\pi P)P = \pi P = \pi$, but it is not unique because the chain Y is not irreducible.

i) Compute $\lim_{n\to\infty} \mathbb{P}(Y_n = 0 \,|\, Y_0 = 0)$.

Answer: $\lim_{n\to\infty} \mathbb{P}(Y_n = 0 \,|\, Y_0 = 0) = \pi_0^{(Y)}$, where $\pi^{(Y)}$ is the unique stationary and limiting distribution of the subchain of Y restricted to the states with even values of i. The computation of $\pi^{(Y)}$ gives $\pi_0^{(Y)} = \frac{1}{2}$.

Exercise 3. (8 points) Let $(X_n,\, n \geq 0)$ be a Markov chain with state space $\mathcal{S} = \{-, 0, +\}$ and transition matrix satisfying

$$p_{+,-} = p_{-,+} = 0, \quad p_{+,0} = p_{-,0} = a \quad \text{and} \quad p_{0,+} = p_{0,-} = b/2$$

where $0 < a, b < 1$.

a) Complete the transition matrix P.

Answer: $P = \begin{pmatrix} 1-a & a & 0 \\ b/2 & 1-b & b/2 \\ 0 & a & 1-a \end{pmatrix}$

b) Compute its eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2$.

Answer: $\mathrm{Tr}(P) = 3 - 2a - b$ and $\det(P) = (1-a)^2\,(1-b) - a\,(1-a)\,b = (1-a)\,(1-a-b)$. Knowing that $\lambda_0 = 1$, we see here that $\lambda_1 = 1-a$ and $\lambda_2 = 1-a-b$ is the solution.

c) Deduce the value of the spectral gap γ.

Answer: $\gamma = \min(a, 2-a-b)$.

d) Under which minimal condition on the parameters a, b does it hold that $\gamma = \lambda_0 - \lambda_1$?

Answer: the condition is $a \leq 2-a-b$, i.e. $2a + b \leq 2$.

Exercise 4. (20 points) We consider the Ising distribution on two vertices $\{1,2\}$. Let $\underline{s} = (s_1, s_2) \in \{-1,+1\}^2$ and

$$\pi_\beta(\underline{s}) = \frac{\exp(\beta s_1 s_2)}{4\cosh(\beta)}$$

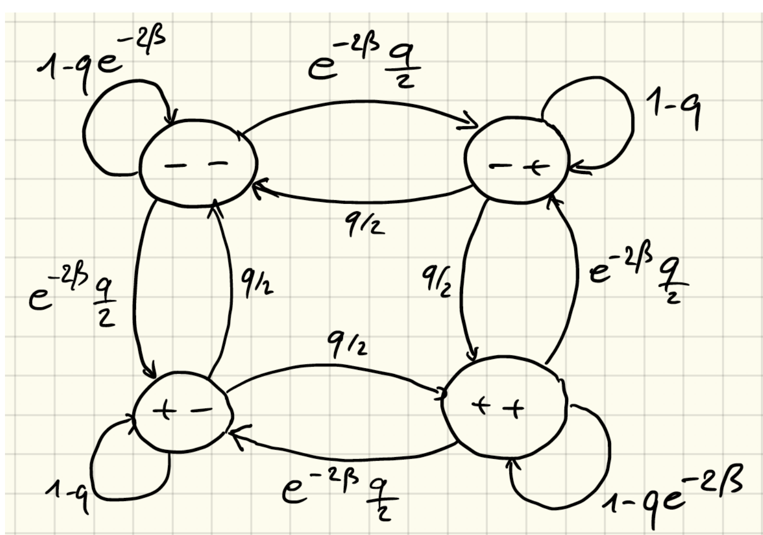
with $\beta \geq 0$. Recall that $\cosh(x) = \frac{1}{2}\,(\exp(x) + \exp(-x))$.

Consider the Metropolis-Hastings Markov chain $(\underline{S}_n,\ n \geq 0)$ defined as follows (starting from the initial uniform distribution on $\{-1,+1\}^2$):

- at step n, select a vertex $v \in \{1,2\}$ uniformly at random;
- propose the move $\underline{S}^{(n)} \to \underline{S}^{(n+1)}$, where $S_v^{(n)}$ is flipped with probability $0 < q \leq 1$;
- accept the move according to the usual Metropolis-Hastings rule.

a) Draw the transition graph of the Markov chain $(\underline{S}_n,\ n \geq 0)$.

Answer: Computing the acceptance probabilities of the Metropolis chain, we obtain:



Consider now the magnetization at time $n \geq 0$:

$$M_n = S_1^{(n)} + S_2^{(n)}$$

b) Explain why the process $(M_n,\, n \geq 0)$ is also a Markov chain.

Answer: The state space of the chain M is $\mathcal{S} = \{-2, 0, +2\}$, which boils down to aggregating the states $(-1,+1)$ and $(+1,-1)$ of the chain $\underline{\mathrm{S}}$ into the single state 0. As seen in class, aggregating two states into one does not preserve in general the Markov property. But here, the incoming and outgoing transition probabilities from states $(-1,+1)$ and $(+1,-1)$ to either state $(-1,-1)$ or state $(+1,+1)$ are the same, so the Markov property is preserved.

c) Compute its state space $\mathcal{S}$ and its transition matrix P.

Answer: The state space of the chain M is $\mathcal{S} = \{-2, 0, +2\}$ and its transition matrix P is given by:

$$P = \begin{pmatrix} 1 - q\ \exp(-2\beta) & q\ \exp(-2\beta) & 0 \\ q/2 & 1-q & q/2 \\ 0 & q\ \exp(-2\beta) & 1 - q\ \exp(-2\beta) \end{pmatrix}$$

d) Show that the chain $(M_n,\, n \geq 0)$ is ergodic for $0 < q \leq 1$ and $0 \leq \beta < +\infty$. Compute its limiting and stationary distribution π.

Answer: M has a finite number of states, is irreducible and aperiodic; it is therefore ergodic [actually NOT when $q = 1$ and $\beta = 0$: this is a mistake in the problem set]. Its limiting and stationary distribution π is given by

$$\pi_0 = \frac{1}{1+\exp(2\beta)} = \frac{\exp(-\beta)}{2\cosh(\beta)} \quad \text{and} \quad \pi_2 = \pi_{-2} = \frac{\exp(2\beta)}{2}\,\pi_0 = \frac{\exp(\beta)}{4\cosh(\beta)}$$

(which by the way does not depend on q).

e) Compute the spectral gap γ of the chain $(M_n,\, n \geq 0)$.

Answer: P has the same structure as in Exercise 3, with $a = q\ \exp(-2\beta)$ and $b = q$, so reusing the result of this exercise, we obtain

$$\gamma = \min(q\exp(-2\beta)), 2 - q\ \exp(-2\beta)) - q)$$

f) When $q = \frac{1}{2}$, deduce a tight upper bound $\|P_0^n - \pi\|_{\mathrm{TV}}$ for $n \geq 1$.

Answer: In this case, $\gamma = \frac{1}{2}\ \exp(-2\beta)$, so

$$\|P_0^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2\sqrt{\pi_0}}\,(1-\gamma)^n = \frac{\sqrt{1+\exp(2\beta)}}{2}\left(1 - \frac{1}{2}\exp(-2\beta)\right)^n$$

g) For what value of q (as a function of β) is the spectral gap γ maximized?

Answer: When $q\ \exp(-2\beta)) = 2 - q\ \exp(-2\beta)) - q$ or $q = 1$, depending on the value of β, i.e.:

$$q = \begin{cases} \frac{2}{1+2\ \exp(-2\beta)} & \text{if } \beta \leq \frac{\ln(2)}{2} \\ 1 & \text{otherwise} \end{cases}$$

Grading scheme

Exercise 1. (12 pts)

each question: 1 pt T/F, 2 pts justif

Exercise 2. (20 pts)

a) 2 pts

b) 4 pts (computation) + 1 pt (detailed balance)

c) 1 pt

d) 1 pt

e) 1 pt

f) 3 pts

g) 2 pts

h) 2 pts

i) 3 pts

Exercise 3. (8 pts)

a) 1 pt

b) 4 pts

c) 2 pts

d) 1 pt

Exercise 4. (20 pts)

a) 5 pts

b) 3 pts

c) 2 pts

d) 3 pts (watch out the small mistake in the problem set)

e) 2 pts

f) 2 pts

g) 3 pts