Homework 6

Exercise 1. The aim of the present exercise is to obtain an exponential bound on the decrease of the total variation distance which is different from the one seen in the lectures. Namely, we will see that for an ergodic Markov chain with state space $\mathcal{S}$, transition matrix P, stationary distribution π and initial distribution $\pi^{(0)} = \delta_i$ (i.e., the chain starts in some fixed state i), we have

$$\|P_i^n - \pi\|_{\mathrm{TV}} \leq C(P)^n \, \|\delta_i - \pi\|_{\mathrm{TV}} \tag{1}$$

where

$$C(P) = \sup_{i \neq k \in \mathcal{S}} \|P_i - P_k\|_{\mathrm{TV}} = \sup_{i \neq k \in \mathcal{S}} \sup_{A \subset \mathcal{S}} (P_i(A) - P_k(A)) = \sup_{i \neq k \in \mathcal{S}} \frac{1}{2} \sum_{j \in S} |p_{ij} - p_{kj}|$$

Remark. We do not assume here that $\mathcal{S}$ is finite nor that detailed balance holds. Besides, please observe that it is always the case that $C(P) \leq 1$, but not necessarily that $C(P) < 1$.

First step. *For any two distributions μ and ν on a common state space $\mathcal{S}$, there always exists a coupling such (X, Y) of μ and ν such that*

$$\|\mu - \nu\|_{\mathrm{TV}} = \mathbb{P}(X \neq Y)$$

a) Define first $\xi_i = \min(\mu_i, \nu_i)$ for $i \in \mathcal{S}$. Note that ξ itself is *not* a distribution, as $\sum_{i \in S} \xi_i \leq 1$ in general. Show that setting $\mathbb{P}(X = Y = i) = \xi_i$ for all $i \in S$ implies indeed that

$$\|\mu - \nu\|_{\mathrm{TV}} = \mathbb{P}(X \neq Y)$$

b) We need now to define $\mathbb{P}(X = i, Y = j)$ for $i \neq j$ so that $\mathbb{P}(X = i) = \mu_i$, $\forall i \in S$ and $\mathbb{P}(Y = j) = \nu_j$, $\forall j \in S$. Show that the following proposal works (it is not the unique one):

$$\mathbb{P}(X = i, Y = j) = \frac{(\mu_i - \xi_i)\,(\nu_j - \xi_j)}{1 - \sum_{k \in S} \xi_k}$$

Second step. *For a given transition matrix P on the state space $\mathcal{S}$, define*

$$\widetilde{C}(P) = \sup_{\substack{\mu, \nu \\ \|\mu - \nu\|_{\mathrm{TV}} > 0}} \frac{\|\mu P - \nu P\|_{\mathrm{TV}}}{\|\mu - \nu\|_{\mathrm{TV}}}$$

(where μ, ν are distributions on $\mathcal{S}$). Then $\widetilde{C}(P) = C(P)$.

c) Show that $\widetilde{C}(P) \geq C(P)$.

d) Show that $\widetilde{C}(P) \leq C(P)$.

Hint. Start by proving that for $i, k \in \mathcal{S}$ and $A \subset \mathcal{S}$, $P_i(A) - P_k(A) \leq C(P)\, 1_{\{i \neq k\}}$, and then consider a coupling (X, Y) of μ and ν and average the above expression over all possible values of $X = i$ and $Y = k$ to obtain an upper bound on

$$\|\mu P - \nu P\|_{\mathrm{TV}} = \sup_{A \subset \mathcal{S}} (\mu P(A) - \nu P(A))$$

Third step. *Conclusion*

e) Prove inequality (1).

Examples.

f) Compute the constant $C(P)$ of the chain of Homework 4, Exercise 2, with state space $\mathcal{S} = \{0, 1\}$ and transition matrix $P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$.

g) Compute the constant $C(P)$ of the chain of Homework 5, Exercise 2, with state space $\mathcal{S} = \{0, 1, 2\}$ and transition matrix $P = \begin{pmatrix} 1-p & p & 0 \\ 1/2 & 0 & 1/2 \\ 0 & p & 1-p \end{pmatrix}$.

h) Compute the constant $C(P)$ of the transition matrix P of the cyclic random walk on the state space $\mathcal{S} = \{0, 1, \ldots, N-1\}$ with N odd ≥ 3, and probabilities p, q of rotating clockwise, resp. counter-clockwise (with $p + q = 1$). Differentiate the case $N = 3$ from the rest. In this case, for what value of p is $C(P)$ the smallest?

Exercise 2. Let $(X_n, n \geq 0)$ be a time-homogeneous Markov chain with state-space $\mathcal{S} = \{0, 1, 2, 3\}$ and let $p \in [0, 1]$. Suppose that the Markov chain evolves according to the transition graph depicted in figure 1.

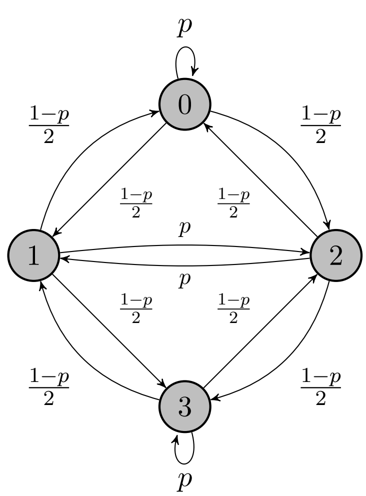

Figure 1: State diagram for the Markov chain $(X_n, n \geq 0)$

a) For which values of p is the chain ergodic? Explain why.

b) Find the stationary distribution π for the values of p found in part a).

c) Show that the chain is reversible.

d) What is the spectral gap γ?

e) Find an asymptotic upper bound on the mixing time $T_\epsilon = \inf\{n \geq 1 : \max_{i \in S} \|P_i^n - \pi\|_{\text{TV}} \leq \epsilon\}$. Given a fixed ϵ, for which value of p is the convergence the fastest?