§ 1.1 Topological manifolds

<u>Recall</u>: A topological space $(X, \tau) =: X$ is said to be

- <u>Hausdorff</u> if given $p, q \in X$ $p \neq q$ there exist $U, V \subseteq X$ s.t. $p \in U$, $q \in V$, $U \cap V = \emptyset$

- <u>second countable</u>: if there exist a <u>countable</u> basis $\mathcal{B}$ for τ

  basis: subfamily of open $\mathcal{B} \subseteq \tau$ s.t. $\forall x \in X$ and $U \ni x$ $\exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U$.

<u>Definition</u>. Let M be a topological space. M is a <u>topological n-manifold</u> if it has the following properties:

1) M is <u>locally Euclidean</u>, i.e. $\forall x \in M$ $\exists$ a neighb. $x \in U \subseteq X$ such that
$$\exists \; U \xrightarrow[\text{homeomorphism}]{\varphi} \hat{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n$$
Remark: choice

2) M is Hausdorff

3) M is second-countable

- We call (U, φ) a coordinate chart — local coordinates: $\varphi(q) = (x_1(q) \dots x_n(q))$
- U coordinate domain / neighbhorood
- φ is centered at p if $\varphi(p) = 0$

~~Examples~~

~~[illegible]~~

Exercise

Show that you can equivalently define manifolds replacing (1) with

- $\forall p \in M$ $\exists p \in U \subseteq M$ and $B_r(x) \subseteq \mathbb{R}^n$

  s.t. $\exists$ $U \xrightarrow[\text{homeomorphism}]{\varphi} B_r(x)$

- " s.t. $U \xrightarrow[\text{homeomorphism}]{\varphi} \mathbb{R}^n$

Important Remarks

$\boxed{1}$ By definition, a topological manifolds has a specific well defined dimension

This follows from (1) and

Theorem (Brouwer's)

If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are homeomorphic

$\Rightarrow$ $\boxed{m = n}$

[2] it is possible to have M which is locally Euclidean but it is not Hausdorff
it is not second countable

Example

(•) Take $X = \coprod_{\mathbb{R}} U$ for $U \subseteq \mathbb{R}^n$ open

this is locally Euclidean and Hausdorff but not second countable

↳ we will see already in the next Lecture why this is important.

(•) Take $X = \{(x,1) \in \mathbb{R}^2\} \sqcup \{(x,-1) \in \mathbb{R}^2\} \big/ \sim$

$(x,1) \sim (x,-1) \iff x \neq 0$

with the quotient topology.

Exercise

X is locally Euclidean and second-countable but it is not Hausdorff

Examples

0) Every open $U \subseteq \mathbb{R}^n$ is a topological n-manifolds

Recall from topology or convince yourself that the properties of being

- Hausdorff
- second-countable

are preserved under passing to subspaces endowed with subspace topology

1) $\boxed{\text{Graphs}}$ $U \subseteq \mathbb{R}^n$ $F : U \to \mathbb{R}^k$ continuos

$$\Gamma(F) = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^k \;\middle|\; \begin{array}{l} x \in U \\ y = F(x) \end{array} \right\}$$

is a n-topological manifold

$$\begin{array}{ccc} \Gamma(F) & \subseteq & \mathbb{R}^n \times \mathbb{R}^k \\ \Big\downarrow{\scriptstyle p_1|_{\Gamma(F)} = \varphi} & & \Big\downarrow{\scriptstyle p_1} \\ U & \subseteq & \mathbb{R}^n \end{array}$$

- φ is continuos with inverse $\varphi^{-1}(x) = (x, F(x))$ which is also continuos

3) Spheres $S^n = \{(x_0, \dots x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \dots + x_n^2 = 1\}$

$\forall\ i = 0, \dots n$ call

$$U_i^{\pm} = \{(x_0, \dots x_n) \in \mathbb{R}^n \mid x_i > 0 \ \ (x_i < 0)\} \subseteq \mathbb{R}^n$$

$\Rightarrow$ • $U_i^+ \cap S^n$ is the graph of a continuos Function, indeed: let $F : \mathbb{B}^n \to \mathbb{R}$, $u \to \sqrt{1 - \|u\|^2}$

$$x_i = \sqrt{1 - (x_0^2 + \dots x_{i-1}^2 + x_{i+1}^2 + \dots + x_n^2)}$$

similarly on • $U_i^- \cap S^n$

$$x_i = -\sqrt{1 - (x_0^2 + \dots x_{i-1}^2 + x_{i+1}^2 \dots + x_n^2)}$$

So, as in Example 2 we have $U_i^{\mp} \cap S^n$ is homeomorphic to $\mathbb{B}^n$

$$\varphi_i^{\pm} : U_i^{\pm} \cap S^n \longrightarrow \mathbb{B}^n$$
$$(x_1, \dots x_{n+1}) \longrightarrow (x_1 \dots x_{i-1}, x_{i+1}, \dots x_n)$$

In the exercises you will be asked to do some more examples

§ 1.2 : Smooth / Differentiable manifolds

<u>Recall</u> : let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$

$$F : U \to V \quad \text{a function}$$

$$(x_1, \dots x_n) \to (F_1(x_1, \dots x_n), \dots F_m(x_1, \dots x_n))$$

F is <u>smooth</u> if each F_i has continuous partial derivatives of all orders

<u>Remark</u>

$\boxed{\text{In this course smooth} = \text{differentiable} = C^\infty}$

We want to be able to say when

$$F : M \to \mathbb{R} \quad \text{is } \underline{\text{smooth}}$$

$$\nearrow$$

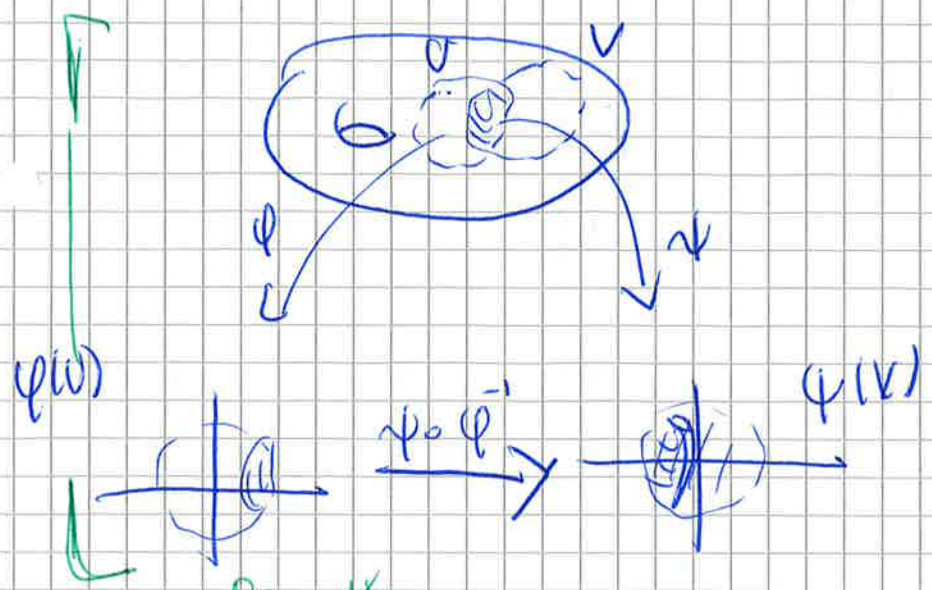
$$F : U$$

By definition $\forall x \in M$ $\exists$ a coordinate chart

$$M \supseteq U \xrightarrow[\text{homeom}]{\varphi} \hat{U} \subseteq \mathbb{R}^n$$

We can look at $F \circ \varphi^{-1} : \hat{U} \to \mathbb{R}$

we know what smooth means



Remark

We want the property of F to be smooth to be $\underline{\text{independent}}$ from the choice of coordinate chart

$\underline{\text{Definition}}$: Let M be an n-topological manifold. Let (U, φ), (V, ψ) two charts s.t. $U \cap V = \emptyset$ we call the composition

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

a $\underline{\text{transition map}}$

We say that (U, φ) and (V, ψ) are $\underline{\text{smoothly compatible}}$ if

$$\boxed{\psi \circ \varphi^{-1} \text{ is a diffeomorphism}}$$

or if $U \cap V = \emptyset$

Definition: A *atlas* for a n-topological manifold M is a collection of charts whose domains cover M

An atlas $\mathcal{A}$ is a *smooth atlas* if any two charts are *smoothly compatible*

Remark
Notice that we already know that $\exists$ inverses of $\psi \circ \varphi^{-1}$, so to show that an atlas is smooth we need to show that all transition maps are smooth $\rightsquigarrow$ diffeomorphism

Def.
We say that an Atlas $\mathcal{A}$ is *maximal* or *complete* if any chart (U, φ) which is smoothly compatible with all charts in $\mathcal{A}$ is already in $\mathcal{A}$.

Def: A *smooth manifold* $(M, \mathcal{A})$ is a pair
- M - n-top. mfd
- $\mathcal{A}$ is a maximal smoothly compatible atlas

We say the $\mathcal{A}$ defines a *smooth structure* or a *differentiable structure* on M

Rmk

A smooth structure is additional data on a given topological manifold there is in general more than one smooth structure

Rmk

Working with maximal atlas is not always convenient. Ideally we want to have as little effort as possible.

So we want a way to speak about the smooth structure $(M, \mathcal{A})$ for any smooth atlas $\mathcal{A}$.

Proposition

(a) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal atlas $\bar{\mathcal{A}}$

$\Rightarrow$ given $(M, \mathcal{A})$ $\exists!$ smooth structure induced by $\mathcal{A}$.

(b) $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ determine the same smooth structure

$\Leftrightarrow$ $(M, \mathcal{A}_1 \cup \mathcal{A}_2)$ is smooth

Proof

(e) let

$\overline{\mathcal{A}} = \{ (U, \varphi)$ chart for $M \mid (U,\varphi)$ smoothly comp. with (U', θ) $\forall (U', \theta) \in \mathcal{A} \}$

$\Rightarrow \mathcal{A} \subseteq \overline{\mathcal{A}}$

We want to show that $\overline{\mathcal{A}}$ is a smooth atlas

$\Leftrightarrow \forall (U, \varphi), (V, \psi) \in \overline{\mathcal{A}}$

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V) \text{ is smooth}$$

Take $x = \varphi(p) \in \varphi(U \cap V)$

since $\mathcal{A}$ is an atlas $\exists (W, \theta) \in \mathcal{A}$ s.t.

$p \in W$. So $p \in W \cap U \cap V$

and By definition of $\overline{\mathcal{A}}$ we have that

$$\varphi \circ \theta^{-1} : \theta(U \cap V \cap W) \to \varphi(U \cap V \cap W)$$
$$\theta \circ \psi^{-1} : \psi(\quad) \to \theta(\quad)$$

are smooth

$\Rightarrow (\varphi \circ \theta^{-1}) \circ (\theta \circ \psi^{-1})$ is smooth

We can repeat the argument at any point of $\varphi(U \cap V)$

$\Rightarrow$ $\boxed{\psi \circ \varphi^{-1} \text{ is smooth}}$

$\Rightarrow$ $\bar{\mathcal{A}}$ is a smooth atlas

[$\bar{\mathcal{A}}$ is the unique maximal atlas containg $\mathcal{A}$

- maximal ~~suppos~~ need to show that if (U, φ) is a smooth chart comp with $\bar{\mathcal{A}}$ $\Rightarrow$ it bel. to $\bar{\mathcal{A}}$. But it is also comp with $(\mathcal{A})$ $\Rightarrow$ it is $\bar{\mathcal{A}}$ by dof
- unique: suppose $\mathcal{B}$ is a max. atlas cont $\mathcal{A} \subseteq \mathcal{B}$ $\Rightarrow$ charts in $\mathcal{B}$ are cpt with charts in $\mathcal{A}$

$$\Rightarrow \mathcal{B} \subseteq \bar{\mathcal{A}} \Rightarrow \mathcal{B} = \bar{\mathcal{A}}$$

$\blacksquare$

(b) If $(M, \mathcal{A}_1 \cup \mathcal{A}_2)$ is smooth

$\Rightarrow$ $\mathcal{A}_1, \mathcal{A}_2$ define the same structure

$$\mathcal{A}_1 \cup \mathcal{A}_2 \text{ smoot} \Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \overline{\mathcal{A}_1 \cup \mathcal{A}_2}$$

$$\mathcal{A}_1 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \overline{\mathcal{A}_1 \cup \mathcal{A}_2}$$

$\Leftarrow$ $\bar{\mathcal{A}}_1 = \bar{\mathcal{A}}_2 \xRightarrow{\mathcal{A}_2}$

$\blacksquare$

given a set, to show that it is possible to give it a structure of smooth manifold, it might be convenient to define

- topology
- transition functions at the same time

Lemma

Let M be a set and $\{U_\alpha\}$ a collection of subset together with maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ s.t.

① $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ is a bijection on a open subset

② $\forall\ \alpha, \beta$ $\varphi_\alpha(U_\alpha \cap U_\beta)$, $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$

③ If $U_\alpha \cap U_\beta \neq \emptyset \Rightarrow \varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth

④ Countably many U_α cover M

⑤ $\forall\ p, q$ either $\exists\ U_\alpha \ni p, q$ or $\exists\ U_\alpha, U_\beta$ s.t. $U_\alpha \ni p$, $U_\beta \ni q$ and $U_\alpha \cap U_\beta = \emptyset$

$\Rightarrow \exists!$ manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart

Proof: Key idea: Define the topology on M saying that a base is given by $\varphi_\alpha^{-1}(V)$, $V \subseteq \mathbb{R}^n$ open ■